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Abstract

The term “belief function” is generally used to refer to a class of capacities that can be viewed as representing ambiguity averse preferences. This paper introduces a definition of equilibrium for normal-form games with ambiguous beliefs, where belief functions are used to describe strategic uncertainty. To capture independence of strategies and beliefs, a novel notion of a “strategic product integral” is introduced for belief functions, based on the Möbius transform of a capacity, and shown to be different from the Choquet integral of an appropriate product capacity. A characterization of the integral in terms of maxmin expected utility expressed relative to elements of the cores of the respective belief functions, is also presented. The resulting equilibrium notion relies on the Möbius transform to embed objectively chosen probabilistic mixed strategies into ambiguous beliefs of opponents about these strategies, while incorporating stronger consistency requirements than those imposed by previous definitions of equilibria under ambiguity.

Keywords: Belief functions; Product capacities; Equilibrium under ambiguity; Strategic uncertainty

JEL Classification: C72; D81

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1 Introduction

Equilibrium concepts under expected utility, such as Nash equilibrium (NE) and its refinements, impose consistency requirements that force players’ equilibrium strategies to be identical to their opponents’ beliefs about these strategies. Hence, in the absence of incomplete information about the game that is played, NE excludes the possibility of any residual strategic uncertainty with respect to player beliefs or predicted outcomes. This could be viewed as a strong assumption in situations where players do not fully understand the environment in which they interact, so that a player or an analyst may want to allow for some strategic uncertainty even in the context of an equilibrium outcome. While the decision theory literature has provided various appealing methods to model choices under uncertainty or ambiguity, introducing such methods to model strategic uncertainty in games requires a reformulation of the usual notion of equilibrium, so that equilibrium strategies are not identical to equilibrium beliefs. Hence, an equilibrium notion that allows for strategic uncertainty must rely on some weakening of the consistency conditions that are usually associated with the idea of equilibrium.

The game theory literature has proposed a number of approaches to defining such weaker notions of equilibrium, based mainly on preferences represented by Choquet expected utility (CEU) (Schmeidler, 1989; Gilboa, 1987; Sarin and Wakker, 1992) or maxmin expected utility (MEU) (Gilboa and Schmeidler, 1989). For example, the notion of “equilibrium under ambiguity” (EuA) introduced in Dow and Werlang (1994), Marinacci (2000) and Eichberger and Kelsey (2000), uses CEU to define an equilibrium as a profile of capacities that describe the players’ beliefs about their opponents’ action choices, where the corresponding consistency requirement restricts the support of each such capacity to pure actions that are optimal given opponents’ beliefs.¹ This results in an equilibrium-in-beliefs notion, which only identifies profiles of pure actions that are consistent with the equilibrium (i.e., the profiles of actions that lie in the support of the capacities that define the equilibrium). Implicitly, this equilibrium definition thus excludes the possibility that players choose mixed strategies—in equilibrium, players are only assumed to play one of the pure actions in their best response sets, following some unspecified procedure. Consequently, the uncertainty in the players’ beliefs, as represented by the equilibrium capacities, is not reflected in a corresponding randomness in predicted outcomes. Furthermore, in games with more than two players, a player may believe that his opponents’ strategies are not independent, and any two players are also allowed to hold inconsistent beliefs about a third player’s strategies.

Alternative approaches based on MEU have been suggested by Klibanoff (1996) and Lo (1996),

¹Defining the support of a capacity is not as straightforward as for additive probability measures—an analysis of various support notions for capacities can be found in Dominiak and Eichberger (2016).
who define equilibrium as a profile of sets of distributions over each player’s opponents’ actions, representing the player’s equilibrium beliefs, augmented in the case of Klibanoff (1996) by a profile of mixed strategies describing actual equilibrium choices. In Klibanoff (1996), the separately specified equilibrium strategies must be optimal given beliefs, and their (independent) products must be contained in the opponents’ belief sets, which, however, need not be restricted to optimal strategies, and may contain elements that allow for correlated actions. Lo (1996) restricts the belief sets to distributions whose marginals only contain optimal mixed strategies, but these distributions need not be consistent or independent across all players. Both these equilibrium notions allow for explicit randomizing using mixed strategies, which, for the case with MEU preferences, can be strictly preferred to pure actions. A problematic aspect of these approaches to defining equilibrium is that while strategies are in some way required to be optimal given beliefs, it is still unclear how the equilibrium belief sets are connected to the actual mixed strategies chosen by the players for Klibanoff (1996), or which of the (optimal) mixed strategies in the equilibrium belief sets are in fact played for Lo (1996).

Lehrer (2012) proposes an equilibrium notion that also allows for strategic uncertainty, but where the ambiguity in beliefs arises directly from actual equilibrium strategies as a consequence of the fact that each player has only partial information regarding the mixed strategies chosen by his opponents, so no additional consistency notion is required. This approach provides a more precise connection between strategies and beliefs than the previously mentioned equilibrium notions, so that equilibrium strategies are automatically contained in belief sets. In this setting, players are allowed to choose mixed strategies, expected payoffs are computed using a version of the concave integral defined by Lehrer (2009), which is, however, shown to possess an equivalent MEU representation, and independence of strategies and beliefs is captured analogously to the independence notion for MEU defined in Gilboa and Schmeidler (1989).

The present paper presents an alternative definition of equilibrium with strategic uncertainty, based on a framework where the uncertainty is represented by capacities. In contrast to the EuA of Dow and Werlang (1994), Marinacci (2000) and Eichberger and Kelsey (2000), players are allowed to explicitly choose mixed strategies, strategies and beliefs are required to satisfy an independence condition, and a consistency notion is introduced that directly incorporates standard mixed strategies as objects of choice into opponents’ beliefs described by capacities. Hence, our proposed equilibrium definition shares some characteristics with the notion of Lehrer (2012), although we use a different modeling framework where the ambiguity can be viewed as arising from a certain incomplete understanding of the strategic environment, which is described by the capacities representing the players’

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2A related approach, introduced by Bade (2011) and Riedel and Sass (2014), allows the players’ objects of choice to include ambiguous randomization devices over their action spaces, which are strategically determined to maximize payoffs. Hence, strategic uncertainty is deliberately introduced through the players’ strategy choices.
There are two key issues that we need to address in order to construct our equilibrium concept: As we do not allow players to choose their strategies by selecting a non-additive or ambiguous randomization device, as in Riedel and Sass (2014) or Bade (2011), we need to find a way to incorporate standard probabilistic mixed strategies into beliefs that are described by capacities. Furthermore, since defining an independent product of capacities is not straightforward, we must also address a more fundamental question of how to define the expectation of a payoff function with respect to a collection of independent capacities.

To tackle both of these problems, we restrict our analysis to a class of capacities usually referred to as “belief functions,” which have initially been analyzed in the works of Dempster (1967) and Shafer (1976). Capacities are normalized monotone set functions that can be interpreted as non-additive probability measures. Belief functions possess an additional monotonicity property that generalizes the standard inclusion-exclusion rules for probability measures. Furthermore, a belief function $\alpha : 2^S \to \mathbb{R}$ over a finite state/action space $S$ is fully characterized by a unique collection of coefficients $m_\alpha(A)$ defined across all non-empty subsets $A \subseteq S$, known as the Möbius transform of $\alpha$, which also constitutes a probability distribution over the non-empty subsets of $S$. Viewing $S$ as the set of actions available to a player in a game, we can then interpret the conditional probability distribution over singleton subsets $\{s\} \subseteq S$ that is associated with the Möbius transform $m_\alpha$, as a mixed strategy of a player who chooses actions from $S$. This interpretation allows us to directly incorporate probabilistic mixed strategies into ambiguous beliefs about such strategies, as described by the belief function $\alpha$. Furthermore, an optimality requirement for such mixed strategies is then also partially consistent with the optimality condition for EuA, when defined based on the notion of support for a capacity introduced by Marinacci (2000), which contains all $s \in S$ for which $m_\alpha(\{s\}) > 0$.

Regarding independence, we also use Möbius transforms as a key building block to introduce a novel notion of an independent product integral for belief functions. While an independent product of two or more capacities can be constructed on rectangular sets by setting the product capacity to be equal to the product of the individual capacities, the non-additivity of capacities as set functions implies that this only pins down the values of the product capacity for rectangular sets, but leaves the values for non-rectangular sets undefined. As a result, independent products of capacities are not uniquely defined, as discussed in Hendon et al. (1996) and Ghirardato (1997). Furthermore, even for a given product capacity, the definition of the Choquet integral can force a certain implicit correlation between the initial capacities over which the product is constructed. This was pointed out by Klibanoff (2001) in the context of an analysis of preferences for randomization, who then concludes that Choquet expected utility is not appropriate to model stochastically independent randomization. For belief functions, a specific type of product called a Möbius product, which is defined through
Möbius transforms, has a uniqueness property as an independent product of belief functions (Ghirardato, 1997, Sect. 3.3), however, as we will show, the Choquet integral of the Möbius product involves similar correlations as those noted in Klibanoff (2001). We will thus argue that the Möbius product is not generally able to capture full independence in a game-theoretic setting.3

The product integral we propose, which we refer to as the strategic product integral (SPI) to reflect its game-theoretic motivation, is constructed as an iterated double expectation, preceded by a minimization operator. When restricted to a one-dimensional setting, the SPI coincides with the Choquet integral. For higher-dimensional environments, the inner expectation is calculated with respect to an independent product of probabilistic mixed strategies, and the outer expectation with respect to an independent product of probability distributions over certain type spaces that are derived from the Möbius coefficients of the belief functions that characterize the uncertainty across the different dimensions. The minimization operator reflects ambiguity aversion by capturing the fact that not all types are known to play unambiguous mixed strategies. When applied across multiple dimensions, the SPI is different from the Choquet integral of an appropriately defined product capacity, in the sense that no product capacity exists whose Choquet integral equals the SPI across all integrable functions. However, while the construction of, and the intuition for the product integral are based on the Möbius transform of belief functions, we show that it can also be characterized as the MEU over a set of priors given by all independent products of elements of the cores of the individual belief functions.4 This MEU characterization of the SPI also shows that the independence notion for belief functions that it characterizes is equivalent to the independence for MEU preferences introduced by Gilboa and Schmeidler (1989), and therefore that the independence captured through the SPI can be viewed as a reinterpretation of Gilboa-Schmeidler independence for settings where uncertainty is described by belief functions.

Based on the proposed product integral, we introduce a notion of equilibrium that is defined by a profile of belief functions, such that the (consistent) beliefs of all opponents of a player $i$ regarding the strategy of $i$ are described by the same belief function $\alpha_i$. In equilibrium, the actual mixed strategy of $i$, which is required to be optimal when $i$’s expected payoffs are computed using the SPI across his mixed strategy and all $\alpha_j$ for $j \neq i$, must be equal to the conditional distribution over $i$’s actions that is induced by the Möbius coefficients for singletons corresponding to $\alpha_i$. The equilibrium consistency condition thus directly incorporates standard mixed strategies into ambiguous beliefs that capture

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3Eichberger and Kelsey (2014) point out that independent beliefs in games could potentially be modeled using the Möbius product, but do not pursue this idea further. Bailey et al. (2005) propose a version of EuA defined by beliefs about opponents’ strategies that are consistent across players, and are modeled as a Möbius product of simple capacities (which are just contractions of probability measures, as defined later in our paper), and apply the resulting equilibrium notion to a public good provision game.

4The core of a capacity is the set of probability measures that weakly dominate the capacity event-wise.
additional strategic uncertainty. The computation of expected payoffs based on the SPI reflects both independence of a player’s own mixed strategies relative to his ambiguous beliefs regarding his opponents’ strategies, and independence across opponents’ strategies. Furthermore, the SPI allows strict preferences for randomization, and hence for mixed strategies, analogous to an MEU setting.\(^5\)

After the first draft of the present paper was completed, the author became aware of a recent working paper, Dominiak and Eichberger (2017), which applies the original notion of EuA to games where the players’ beliefs are represented by a convex combination of a belief function and its dual, and thus allows for beliefs that combine various attitudes towards optimism and pessimism. In their paper, the Möbius transforms corresponding to equilibrium beliefs are constructed based on common probability distributions over action spaces that are used to define the conditional Möbius weights of singleton sets, together with player-specific weights for non-singleton sets. Hence, consistency of the common probability distributions is required in the associated EuA, which is analogous to the consistency of the players’ mixed strategies in the equilibrium notion we introduce. However, in contrast to our paper, Dominiak and Eichberger (2017) maintain the original assumptions of EuA (so inconsistencies in the overall beliefs are still possible), expected utilities are defined using the Choquet integral, and the resulting equilibrium definition yields an equilibrium-in-beliefs notion that does not allow for explicit randomizations.

The paper is structured as follows: Section 2 presents the basic definitions of capacities, belief functions and Möbius transforms. The strategic product integral is introduced and characterized in Section 3. Section 4 presents and analyzes our equilibrium notion for normal-form games, and Section 5 concludes the paper and discusses possible extensions.

## 2 Capacities and belief functions

Given our objective of using capacities to model beliefs in finite games, we will restrict our analysis to capacities defined on finite sets. A capacity on a finite set \(S\) is a set function \(\alpha : 2^S \to \mathbb{R}\) that satisfies

\[
(i) \quad \alpha(\emptyset) = 0, \quad \alpha(S) = 1, \text{ and }
\]

\[
(ii) \quad \text{for all } A, B \subseteq S, A \subseteq B \Rightarrow \alpha(A) \leq \alpha(B).
\]

Thus, property (i) implies that \(\alpha\) is normalized to values between \(0\) and \(1\), and property (ii) that it is monotone with respect to set inclusion. \(\alpha\) is convex if it also satisfies that

\[
(iii) \quad \text{for all } A, B \subseteq S, \quad \alpha(A \cup B) + \alpha(A \cap B) \geq \alpha(A) + \alpha(B),
\]

\(^5\)This holds even when the realizations of a corresponding randomization device are included in a Savage-style state space, in which case Eichberger and Kelsey (1996) have shown that such randomizations can never be strictly preferred if expected payoffs are defined by the Choquet integral of an appropriate product capacity.
and is *totally monotone* if

(iv) for every $n > 0$ and every collection $A_1, \ldots, A_n \subseteq S$,

\[
\alpha \left( \bigcup_{i=1}^{n} A_i \right) \geq \sum_{\varnothing \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \alpha \left( \bigcap_{i \in I} A_i \right).
\]

Note that property (iv) is just a version of the so-called exclusion-inclusion rules for probability measures, where the equality is replaced with a weak inequality. A capacity that satisfies this property is called a *belief function*. A probability measure is thus a belief function for which (iv), and thus (iii), hold with equality.

The most common notion of integration for capacities is the Choquet integral (Choquet, 1953). If $\alpha$ is a capacity on a finite set $S = \{s_1, \ldots, s_n\}$, and $u : S \to \mathbb{R}$ is such that $u(s_1) \geq u(s_2) \geq \cdots \geq u(s_n)$, then the *Choquet integral* of $u$ can be defined as

\[
\int_{S} u(s) d\alpha(s) = \sum_{i=1}^{n-1} [u(s_i) - u(s_{i+1})] \alpha(\{s_1, \ldots, s_i\}) + u(s_n)
\]

\[
= \sum_{i=1}^{n} u(s_i) \alpha(\{s_1, \ldots, s_i\}) - \alpha(\{s_1, \ldots, s_{i-1}\})]
\]

Thus, if a player’s beliefs are defined by a capacity $\alpha$, we can interpret the Choquet integral of a utility function $u$ as an expected utility, usually referred to as the Choquet expected utility (CEU).

An interesting and useful property of capacities on a finite set $S$ is that there exists a finite set of capacities that form a linear basis for all capacities on $S$. If we let $\mathcal{S}$ denote the set of all non-empty subsets of $S$, this basis is defined by a family of capacities $\{v_A\}_{A \in \mathcal{S}}$, where for every $B \in \mathcal{S}$,

\[
v_A(B) = \begin{cases} 
1, & \text{if } B \supseteq A, \\
0, & \text{otherwise}.
\end{cases}
\]

Then, for every capacity $\alpha$ on $S$, there exist unique coefficients $\{m_{\alpha}(A)\}_{A \in \mathcal{S}}$ such that

\[
\alpha = \sum_{A \in \mathcal{S}} m_{\alpha}(A) v_A,
\]

which implies that for all $A \in \mathcal{S}$,

\[
\alpha(A) = \sum_{B \subseteq A} m_{\alpha}(B).
\]

This decomposition of $\alpha$ is known as the *Möbius transform* of $\alpha$. It was first derived by Shapley (1953) in the context of cooperative games, and used to characterize capacities by Dempster (1967) and Shafer (1976).

If $\alpha$ is a belief function, it is well-known (see Shafer, 1976; Chateauneuf and Jaffray, 1989; Gilboa and Schmeidler, 1994) that $m_{\alpha}(A) \geq 0$ for all $A \in \mathcal{S}$, and that $m_{\alpha}$ defines a probability measure on $\mathcal{S}$.
Furthermore, Gilboa and Schmeidler (1994) show that the Choquet integral with respect to a belief function $\alpha$ can be expressed as

$$\int_S u(s)d\alpha(s) = \sum_{A \in \mathcal{S}} m_\alpha(A)\left[\min_{s \in A} u(s)\right],$$

so the integral is equivalent to an average of minima of the integrand $u$. Convex capacities, and hence belief functions, which are convex, are known to correspond to ambiguity/uncertainty averse preferences (see Schmeidler, 1989, pp. 582–583). Hence, the above expression for CEU can be viewed as representing ambiguity aversion by summing over products of probabilities of observing particular sets $A \in \mathcal{S}$, times the minimum/worst-case utilities attained by $u$ on these sets.

Even though we have introduced the M"obius transform as a mathematical characterization of a capacity, for belief functions, the M"obius transform possesses an alternative foundation based on the "theory of evidence" introduced by Dempster (1967), and further developed by Shafer (1976). Following this interpretation, $m_\alpha(A)$ can be viewed as capturing the extent to which available evidence or information supports the complete event $A$, and which cannot be further sub-divided toward subsets of $A$. Such an interpretation may be of particular interest for game-theoretic applications, if an equilibrium is seen as reflecting information (possibly based on historical or institutional sources) regarding the possible actions of a player in a particular strategic setting. The weights assigned by $m_\alpha$ to singleton sets could then be interpreted as capturing the opponents’ beliefs that a player assigns probability directly to singleton actions, as in a mixed strategy, whereas the weights assigned by $m_\alpha$ to any non-singleton set $A$, could represent the opponents’ beliefs that the player does not choose a particular (mixed or pure) strategy, but arbitrarily chooses an action from the set $A$, such as, for example, when $A$ is the set of undominated actions and the player is only known to not play any dominated actions. This interpretation of the M"obius transform $m_\alpha$ as reflecting probabilistic evidence regarding the actions of a player in a game, also illustrates one advantage of restricting our analysis to belief functions, as the values of the M"obius transform are not guaranteed to be non-negative for more general capacities.

### 3 The strategic product integral

#### 3.1 A motivating example: Simple capacities

The motivation for the methods proposed in this paper can be illustrated using a class of belief functions usually referred to as simple capacities. A simple capacity is essentially a contraction of an additive probability measure on all strict subsets of $S$. Such a capacity $\alpha$ is defined by a probability

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6 This characterization of the Choquet integral, as well as the Möbius decomposition of a capacity, can also be generalized to infinite $S$, as shown in Gilboa and Schmeidler (1995).
measure \( \pi \in \Delta S \) and a number \( \delta \in (0, 1) \), so that
\[
\alpha(A) = \begin{cases} 
(1 - \delta)\pi(A), & \text{for all } A \subseteq S, \\
1, & \text{for } A = S.
\end{cases}
\]
The Möbius transform of a simple capacity \( \alpha \) is given by
\[
m_\alpha(\{s\}) = (1 - \delta)\pi(s)
\]
for all singleton subsets \( \{s\} \in \mathcal{S} \),
\[
m_\alpha(A) = 0
\]
for all non-empty strict subsets \( A \) of \( S \), such that \( A \) is not a singleton and \( A \neq S \), and
\[
m_\alpha(S) = \delta.
\]
Simple capacities have an attractive interpretation, essentially due to Ellsberg (1961).\(^7\) The probability measure \( \pi \) can be interpreted as a best estimate for the distribution of states in \( S \), and \((1 - \delta)\) as a degree of confidence in the distribution given by \( \pi \). Hence, \( \delta \) can be viewed as a degree of ambiguity, or as a probability that randomness in \( S \) cannot be predicted, such as in a case where the choice of a state in \( S \) is made by an irrational opponent in a game. The CEU of a function \( u : S \to \mathbb{R} \) with respect to a simple capacity \( \alpha \), given by
\[
\int_S u(s) d\alpha(s) = (1 - \delta) \sum_{s \in S} \pi(s) u(s) + \delta \min_{s \in S} u(s),
\]
supports this interpretation, as it is just a weighted average of the expectation of \( u \) under \( \pi \), with weight \((1 - \delta)\), and the minimum of \( u \) over \( S \), with weight \( \delta \). Thus, the ambiguity aversion inherent in \( \alpha \) is reflected by the fact that with probability \( \delta \), no prediction is possible, and hence, in this case, the worst-case outcome defines the value of the integral.

Consider now a strategic game where the beliefs of a player regarding his opponents’ actions are represented by simple capacities. For a given player, assume first that he has only one opponent, denoted by 1, let the action set of this opponent be given by \( S_1 = \{a, b\} \), and let \( u : S_1 \to \mathbb{R} \) be the initial player’s payoffs corresponding to a fixed own action. (The player’s own actions will be irrelevant for the following argument, so are suppressed for now.) If the player’s beliefs regarding opponent 1’s actions are given by a simple capacity \( \alpha_1 \) with additive part \( \pi_1 \) and degree of ambiguity \( \delta_1 \), his CEU is given by
\[
\int_{S_1} u d\alpha_1 = (1 - \delta_1) [\pi_1(a) u(a) + \pi_1(b) u(b)] + \delta_1 \min\{u(a), u(b)\}.
\]
\(^7\)The interpretation in Ellsberg (1961) was, however, not framed in the language of capacities. Hence, the axiomatization of preferences represented by simple capacities that was derived in a more general setting by Eichberger and Kelsey (1999), also provides a decision-theoretic foundation for Ellsberg’s interpretation.
As noted before, we can interpret $\delta_1$ as the probability that opponent 1 is irrational, in which case no predictions regarding his actions are possible. The additive part of $\alpha_1$ can then be interpreted as the best estimate regarding opponent 1’s mixed strategy in case this opponent is rational. If we thus allow the opponent’s strategy space to include mixed strategies, then even an irrational opponent may end up playing an (arbitrary, unpredictable) mixed strategy $\sigma_1 \in \Delta S_1$, so he could be viewed as playing an ambiguous strategy defined by the set of all mixed strategies $\Delta S_1$. Using this notation and interpretation, the CEU can equivalently be rewritten as

$$\int_{S_1} u d\alpha_1 = (1 - \delta_1)[\pi_1(a)u(a) + \pi_1(b)u(b)] + \delta_1 \min_{\sigma_1 \in \Delta S_1} [\sigma_1(a)u(a) + \sigma_1(b)u(b)].$$

Hence, the CEU is a weighted average of an integral and a minimum over integrals.

Next, consider a game where a given player has two opponents, 1 and 2, with action sets $S_1 = \{a, b\}$ and $S_2 = \{x, y\}$, and a payoff function $u : S_1 \times S_2 \to \mathbb{R}$ corresponding to a fixed own action. Assume that the two opponents act independently, and that the player under consideration has independent beliefs regarding the strategies of these opponents that are represented by simple capacities $\alpha_1$ and $\alpha_2$, with additive parts $\pi_1$ and $\pi_2$, and degrees of ambiguity $\delta_1$ and $\delta_2$, respectively.

Extending the Ellsberg interpretation of CEU to the expected utility over the uncertainty induced by the strategies of the two opponents, requires a notion of an independent product of the two capacities $\alpha_1$ and $\alpha_2$. As recognized in the literature on capacities (Hendon et al., 1996; Ghirardato, 1997), independent products of capacities are not unique. However, Ghirardato (1997) shows that for belief functions there is a unique independent product belief function that also possesses a Fubini-type property. This product is defined through products of Möbius transforms, and is usually referred to as a Möbius product. For the example we consider, the Möbius product $\alpha$ of the capacities $\alpha_1$ and $\alpha_2$, is defined by specifying for any $A \subseteq S_1 \times S_2$, the Möbius transform of the product capacity, $m_\alpha$, as

$$m_\alpha(A) = \begin{cases} m_{\alpha_1}(A_1)m_{\alpha_2}(A_2), & \text{if } A = A_1 \times A_2 \text{ with } A_1 \subseteq S_1 \text{ and } A_2 \subseteq S_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the Möbius transform of the product $\alpha$ can only be positive for sets $A$ that are rectangles in $S_1 \times S_2$.

Given the Möbius transform $m_\alpha$, it is then straightforward to compute the CEU with respect to the Möbius product $\alpha$, based on the expression for the Choquet integral from equation (1). Letting the expected utility of $u$ with respect to the product measure of the additive parts $\pi_1$ and $\pi_2$ be denoted
by \(E_{\pi_1 \times \pi_2}[u]\), the CEU can be written as

\[
\int_{S_1 \times S_2} u \alpha = (1 - \delta_1)(1 - \delta_2)E_{\pi_1 \times \pi_2}[u] \\
+ (1 - \delta_1)\delta_2 [\pi_1(a) \min\{u(a,x), u(a,y)\} + \pi_1(b) \min\{u(b,x), u(b,y)\}] \\
+ \delta_1 (1 - \delta_2) [\pi_2(x) \min\{u(a,x), u(b,x)\} + \pi_2(y) \min\{u(a,y), u(b,y)\}] \\
+ \delta_1 \delta_2 \min\{u(a,x), u(a,y), u(b,x), u(b,y)\}.
\]

If we now try to apply Ellsberg’s interpretation to the CEU with respect to the Möbius product, we can take a Bayesian perspective where in a two-stage process, the types of the two opponents are determined in stage one, based on an independent product of the probability measures \((1 - \delta_1, \delta_1)\) and \((1 - \delta_2, \delta_2)\), and the mixed or ambiguous strategies of the respective types are chosen subsequently in stage two. We can then view \(\delta_1\) and \(\delta_2\) as probabilities that opponents 1 and 2, respectively, are irrational, so \((1 - \delta_1)(1 - \delta_2)\) is the probability that both opponents are rational, \((1 - \delta_1)\delta_2\) the probability that opponent 1 is rational and 2 is irrational, and so on. Similarly, \(\pi_1\) and \(\pi_2\) can be interpreted as the mixed strategies chosen by the rational types of the two opponents.

Note, however, that in the case where 1 is rational and 2 is not, the expression for the CEU would imply that an irrational opponent 2 may choose a worst-case action conditional on 1’s realized action, even when 1 randomizes according to \(\pi_1\), which indicates that the strategy of an irrational opponent 2 can be correlated with \(\pi_1\). Furthermore, when both opponents are irrational (which occurs with probability \(\delta_1 \delta_2\)), the expression for the CEU also suggests that they can perfectly correlate their actions to attain the worst-case outcome according to \(u\). Thus, while the rational types choose given mixed strategies \(\pi_1\) and \(\pi_2\) without awareness of the other opponent’s rationality, irrational types are implicitly assumed to know whether the other player is rational or irrational, and to be able to coordinate their action choices with both the rational and irrational types of the other player.

This suggests that the Möbius product, in conjunction with CEU, describes an environment where the irrational opponents are in some way omniscient, so they have complete information about all players’ types, and can coordinate with both the rational and irrational types of the other player. However, if an assumption that players choose their actions independently in a strategic setting is maintained for such a Bayesian framework, then even an irrational opponent should not have additional information regarding the other opponent’s rationality (beyond what a rational player knows), and should only be able to choose his action once, without the ability to coordinate/correlate with other players. Hence, while the Möbius product is easy to define and apply to calculate the resulting CEU, it is difficult to interpret in a strategic setting with full independence when capacities are used to describe the strategies of opponents, and thus does not seem to provide an appropriate method to model independent beliefs in such games.
We now propose a notion of a product integral for independent simple capacities based on the Ellsberg interpretation of the Choquet integral for the case of a single opponent, combined with the two-stage Bayesian setup we employed to interpret the integral with respect to the Möbius product across two opponents. We would like such a product integral to be consistent with a two-stage process where in stage one, the types of the two opponents are determined based on the independent product of the probability measures \((1 - \delta_1, \delta_1)\) and \((1 - \delta_2, \delta_2)\), and in stage two, the respective realized types choose independent strategies given by the mixed strategies \(\pi_1\) and \(\pi_2\) for rational types, and by the ambiguous strategies \(\Delta S_1\) and \(\Delta S_2\) for irrational types. In addition, the ambiguity aversion of the initial player is reflected by the fact that he considers the worst-case outcome as a function of the ambiguous choices of irrational opponents. The resulting product integral generalizes the expression from equation (2) for the CEU in the one-opponent case, in the sense that it is a weighted sum of product integrals with respect to additive probability measures, combined with a minimization over the ambiguous mixed-strategy choices of irrational opponents.

To define such a strategic product integral with respect to simple capacities \(\alpha_1\) and \(\alpha_2\), let \(\sigma_i\) denote a mixed strategy for the irrational type of opponent \(i \in \{1, 2\}\) that is assumed to be chosen arbitrarily from \(\Delta S_i\), and recall that \(\pi_1\) and \(\pi_2\) denote the additive components of \(\alpha_1\) and \(\alpha_2\). The strategic product integral of a function \(u : S_1 \times S_2 \to \mathbb{R}\) with respect to the capacities \(\alpha_1\) and \(\alpha_2\) is then defined by

\[
\int_{S_1 \times S_2} ud(\alpha_1 \boxtimes \alpha_2) := \min_{\sigma_1 \in \Delta S_1, \sigma_2 \in \Delta S_2} \left\{ (1 - \delta_1)(1 - \delta_2)E_{\pi_1 \times \pi_2}[u] + (1 - \delta_1)\delta_2E_{\pi_1 \times \sigma_2}[u] \right. \\
+ \left. \delta_1(1 - \delta_2)E_{\sigma_1 \times \pi_2}[u] + \delta_1\delta_2E_{\sigma_1 \times \sigma_2}[u] \right\}.
\]  

(3)

The interpretation of this integral is straightforward: The probabilities of the various combinations of rational and irrational opponents are determined independently by a standard product measure, and conditional on each of these combinations, the expected utility is given by an expectation with respect to product measures of additive probabilities defined by mixed strategies. The minima over \(\sigma_1\) and \(\sigma_2\) reflect the fact that the strategies of irrational opponents are unpredictable, and that the player whose payoffs are considered is ambiguity averse.\(^8\)

### 3.2 Main definition of the strategic product integral

The strategic product integral introduced in the previous section for the product of two simple capacities, can be generalized to a product of any finite number \(n\) of belief functions. Let \(\alpha_i\), for \(i \in \{1, \ldots, n\}\), denote a belief function \(\alpha_i : 2^{S_i} \to \mathbb{R}\) defined on a finite set \(S_i\), and let \(\mathcal{S}_i\) denote the set of all non-empty subsets of \(S_i\). Furthermore, let \(\mathcal{S}_{i}^s \subset \mathcal{S}_i\) denote the set of all singleton subsets of \(S_i\). Then

\(^8\)While the paper is restricted to ambiguity averse decision-makers, the concluding section indicates how the product integral can easily be extended to situations where preferences combine various attitudes towards optimism or pessimism.
there exists a collection of Möbius coefficients \((m_{a_i}(A_i))_{A_i \in \mathcal{S}_i}\) that also defines a probability measure on \(\mathcal{S}_i\), and based on which the Choquet integral with respect to \(a_i\) of any function \(u : S_i \rightarrow \mathbb{R}\) can be calculated as

\[
\int_{S_i} u(s_i)da_i(s_i) = \sum_{A_i \in \mathcal{S}_i} m_{a_i}(A_i)[\min_{s_i \in A_i} u(s_i)].
\]

If we consider a game-theoretic application, we can interpret the capacity \(a_i\) as describing the beliefs of any player other than \(i\) regarding the actions of player \(i\), and view the situation where \(i\) can assign precise weights according to \(m_{a_i}\) to singleton actions, as a reflection of rationality for \(i\).\(^9\) Hence, if we define

\[
\rho_{a_i} := \sum_{\{s_i\} \in \mathcal{S}_i^a} m_{a_i}(\{s_i\}),
\]

then \(\rho_{a_i}\) can be interpreted as the probability assigned by the opponents of player \(i\) to a rational type of \(i\), and \(\pi_{a_i} = (\pi_{a_i}(s_i))_{s_i \in S_i}\), with

\[
\pi_{a_i}(s_i) := \frac{m_{a_i}(\{s_i\})}{\rho_{a_i}},
\]

as the mixed strategy played by such a rational type.\(^{10}\) Furthermore, the weight assigned by \(m_{a_i}\) to any non-singleton set \(A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^a\) can be interpreted as describing additional beliefs regarding the probability of an alternative type of \(i\), who makes his choice from \(A_i\) based on some arbitrary procedure, including a possibly mixed strategy. Hence, we will identify any such non-singleton set \(A_i\) with a type of player \(i\) whose strategy is defined by the set \(\Delta A_i\) of probability distributions with support in \(A_i\), and refer to such a type as a strategic type \(A_i\). For example, if the full complementary probability \(1 - \rho_{a_i}\) is assigned to the set of all actions \(S_i\), such as in the case when \(a_i\) is a simple capacity, then the only alternative strategic type is the irrational type \(S_i\), who chooses an arbitrary mixed strategy. Other possible strategic types might correspond to the subsets of \(S_i\) containing all actions that are not strictly dominated, or those that are rationalizable.

Before defining the strategic product integral, note that given any belief function \(a_i\), we can consider a set of strategic types \(\Theta_i := \{\mathcal{S}_i^a\} \cup (\mathcal{S}_i \setminus \mathcal{S}_i^a)\), with an arbitrary element denoted by \(\theta_i\), where \(\mathcal{S}_i^a\) is the rational type, who is assigned probability \(\rho_{a_i}\) according to \(a_i\), and any other type \(A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^a\) is an alternative strategic type, who is assigned probability \(m_{a_i}(A_i)\).\(^{11}\) The Möbius coefficients \((m_{a_i}(A_i))_{A_i \in \mathcal{S}_i}\) thus induce a probability distribution \(p_{a_i}\) over the type set \(\Theta_i\) that is defined

\(^9\)Note that ambiguous strategies defined by specific imprecise randomization devices, as considered by Riedel and Sass (2014), are not allowed as payoff-maximizing objects of choice for any type of player.

\(^{10}\)Defining \(\pi_{a_i}\) in this way requires of course that \(\rho_{a_i} > 0\), if \(\rho_{a_i} = 0\), \(\pi_{a_i}\) can be set to be any arbitrary distribution over \(S_i\), and all our subsequent definitions and results still apply.

\(^{11}\)Note the distinction between the set \(\mathcal{S}_i^a\), whose elements are the singleton subsets of \(S_i\), and the set \(\{\mathcal{S}_i\}\), which contains the single element \(\mathcal{S}_i^a\).
by

\[ p_{\alpha_i}(\theta_i) := \begin{cases} 
\rho_{\alpha_i} & \text{if } \theta_i = \{\mathcal{S}_i^g\}, \\
m_{\alpha_i}(A_i) & \text{if } \theta_i = A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g.
\end{cases} \]

The Choquet integral with respect to \( \alpha_i \) of any function \( u : S_i \to \mathbb{R} \) can then be expressed as

\[
\int_{S_i} u(s_i) d\alpha_i(s_i) = \rho_{\alpha_i} \sum_{s_i \in S_i} \pi_{\alpha_i}(s_i) u(s_i) + \sum_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} p_{\alpha_i}(A_i) \left[ \min_{s_i \in A_i} u(s_i) \right]
\]

\[ = \rho_{\alpha_i} \sum_{s_i \in S_i} \pi_{\alpha_i}(s_i) u(s_i) + \sum_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} p_{\alpha_i}(A_i) \left[ \min_{\sigma_{A_i} \in \Delta A_i} \sum_{s_i \in A_i} \sigma_{A_i}(s_i) u(s_i) \right]. \]

Furthermore, if we let \( \Delta_i := \times_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} \Delta A_i \), with arbitrary element \( \sigma_i = (\sigma_{A_i})_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} \), and define

\[
\langle \pi_{\alpha_i}^{\theta_i} | \sigma_i \rangle := \begin{cases} 
\pi_{\alpha_i} & \text{if } \theta_i = \{\mathcal{S}_i^g\}, \\
\sigma_{A_i} & \text{if } \theta_i = A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g,
\end{cases}
\]

then the Choquet integral can be rewritten as

\[
\int_{S_i} u(s_i) d\alpha_i(s_i) = \min_{\sigma_i \in \Delta_i} \left\{ \sum_{\theta_i \in \Theta_i} p_{\alpha_i}(\theta_i) \mathbb{E}_{\langle \pi_{\alpha_i}^{\theta_i} | \sigma_i \rangle} [u] \right\} = \min_{\sigma_i \in \Delta_i} \left\{ \mathbb{E} p_{\alpha_i} \left[ \mathbb{E}_{\langle \pi_{\alpha_i}^{\theta_i} | \sigma_i \rangle} [u] \right] \right\}. \tag{4}
\]

Thus, the Choquet integral of \( u \) with respect to \( \alpha_i \) is equivalent to a double expectation with respect to additive probabilities—an outer expectation with respect to the distribution over types \( p_{\alpha_i} \), and for each type \( \theta_i \), an inner expectation with respect to the distribution over states/actions \( s_i \) given by \( \pi_{\alpha_i} \) or \( (\sigma_{A_i})_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} \), respectively—combined with a minimization over \( \sigma_i = (\sigma_{A_i})_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} \).\(^{12}\)

To define the strategic product integral of a function \( u : \times_i S_i \to \mathbb{R} \) with respect to the collection of independent belief functions \((\alpha_i)_{i \in I}\), let \( \Theta := \times_i \Theta_i \), let \( p \in \Delta \Theta \) denote the independent product probability measure of \((p_{\alpha_i})_{i \in I}\), so \( p := p_{\alpha_1} \times \cdots \times p_{\alpha_n} \), and for each \( \theta \in \Theta \), let \( \langle \pi^{\theta} | \sigma \rangle \) denote the independent product probability measure of \((\langle \pi_{\alpha_i}^{\theta_i} | \sigma_i \rangle)_{i \in I}\), so \( \langle \pi^{\theta} | \sigma \rangle := \langle \pi_{\alpha_1}^{\theta_1} | \sigma_1 \rangle \times \cdots \times \langle \pi_{\alpha_n}^{\theta_n} | \sigma_n \rangle \). Then the strategic product integral (SPI) is defined by

\[
\int_{\times_i S_i} u d(\bigotimes \alpha_i) := \min_{\sigma \in \times_i \Delta_i} \left\{ \mathbb{E}_p \left[ \mathbb{E}_{\langle \pi^{\theta} | \sigma \rangle} [u] \right] \right\}.
\]

Hence, the outer expectation is taken over profiles of types \( \theta = (\theta_1, \ldots, \theta_n) \) with respect to the product distribution \( p \), and the inner expectation is taken over state/action profiles \( s = (s_1, \ldots, s_n) \) for every profile \( \theta \), with respect to the product distribution \( \langle \pi^{\theta} | \sigma \rangle \). The resulting double expectation is then minimized over the distribution profiles \( \sigma \in \times_i (\times_{A_i \in \mathcal{S}_i \setminus \mathcal{S}_i^g} \Delta A_i) \) associated with non-rational strategic

\(^{12}\)Note that this double expectation is different from a double integral of a function defined over two variables, since here the function \( u \) that is integrated only depends on the “inner” variable of integration \( s_i \), whereas the “inner” distribution \( \langle \pi_{\alpha_i}^{\theta_i} | \sigma_i \rangle \) depends on the variable of integration for the “outer” integral, \( \theta_i \).
types. It is important to note here that while the one-dimensional SPI is equivalent to the Choquet integral (4), the $n$-dimensional SPI is not defined as a Choquet integral, and, as shown later, is in general not equivalent to the Choquet integral of any capacity.

### 3.3 A characterization based on maxmin expected utility

In this section, we present a characterization of the SPI based on maxmin expected utility. Maxmin expected utility (MEU) provides an alternative representation for ambiguity-averse preferences, where a decision-maker’s objective is to maximize the worst-case expected utility attained over a convex set of probability measures/priors. Axiomatic foundations for such a representation have been provided by Gilboa and Schmeidler (1989) for environments with subjective uncertainty, and by Gajdos et al. (2008) for environments with objective uncertainty. The principal connection between MEU and CEU was pointed out by Schmeidler (1989), and relies on the notion of a core of a capacity, which is defined for any capacity $\alpha : 2^S \to \mathbb{R}$, by

$$C(\alpha) := \{ q \in \Delta S \mid q(A) \geq \alpha(A) \text{ for all } A \subseteq S \}.$$  

Hence, the core of $\alpha$ is the set of all probability measures that weakly dominate $\alpha$ event-wise.\footnote{The core of a capacity is also related to various notions of support for capacities, as analyzed in Dominiak and Eichberger (2016). In particular, Dominiak and Eichberger (2016) show that for convex capacities, the intersection of the support across all probability measures in the core of a convex capacity, is equal to the support of this capacity as defined by Marinacci (2000), i.e., to the set of all singletons with strictly positive Möbius weight.}

Then, if $\alpha$ is a convex capacity, its core is always non-empty (Shapley, 1971), and Schmeidler (1986) shows that for any $u : S \to \mathbb{R}$,

$$\int_S u d\alpha = \min_{q \in C(\alpha)} \int_S u dq. \quad (5)$$

As in the definition of the SPI, consider a collection of belief functions $\alpha_i : 2^{S_i} \to \mathbb{R}$ for $i \in \{1, \ldots, n\}$, and a function $u : \times_i S_i \to \mathbb{R}$. Since belief functions are convex, their cores are non-empty, and we can define a set of product probability measures on $\times_i S_i$ generated by taking independent products of core elements $q_i \in C(\alpha_i)$, by setting

$$\otimes_i C(\alpha_i) := \{ q_1 \times \cdots \times q_n \mid q_i \in C(\alpha_i) \text{ for each } i \in \{1, \ldots, n\} \}.$$  

Using an Anscombe-Aumann horse-race-lotteries framework, Gilboa and Schmeidler (1989) construct a preference relation on a product space of states of nature (where each individual state space maps into the same space of horse-race lotteries and is endowed with a preference relation represented by a set of priors following MEU), by considering the MEU over the set of product measures generated by the independent products of the individual priors corresponding to each element in the product space. Gilboa and Schmeidler (1989, pp. 150-152) then show that the induced preference
relation is the unique preference relation on the product space that satisfies their initial axioms and some behavioral independence properties. Based on the Gilboa-Schmeidler independence requirements, we can consider a preference relation over products of belief functions \( \alpha_i \) that is induced by payoffs

\[
\min_{q \in \bigotimes_{i \in S_i} C(\alpha_i)} \int_{x \in S_i} udq.
\]

This preference relation then turns out to be equivalent to the one defined by the SPI, as shown by the following characterization in terms of MEU, which implies that the independence requirements of the SPI are equivalent to those introduced by Gilboa and Schmeidler (1989), and therefore that the notion of independence defined by the SPI can be viewed as a reinterpretation of Gilboa-Schmeidler independence in the context of uncertainty represented through independent belief functions.\(^{14}\)

**Proposition 1.** For any collection of belief functions \( \alpha_i : 2^{S_i} \to \mathbb{R} \) such that \( i \in \{1, \ldots, n\} \) and each \( S_i \) is a finite set, and any \( u : x \in S_i \to \mathbb{R} \),

\[
\int_{x \in S_i} ud(\pi|_{\alpha_i}) = \min_{q \in \bigotimes_{i \in S_i} C(\alpha_i)} \int_{x \in S_i} udq.
\]

*Proof.* The proof follows mainly from a characterization of the core of belief functions derived in Chateauneuf and Jaffray (1989). Considering any capacity \( \alpha_i : 2^{S_i} \to \mathbb{R} \), and using our notation, their Proposition 5 shows that for every \( q_i \in C(\alpha_i) \), there exists a collection \( (\sigma_{A_i})_{A_i \in S_i \setminus S_i} \times A_i \in S_i \setminus S_i \Delta A_i \), such that for each \( s_i \in S_i \),

\[
q_i(s_i) = m_{\alpha_i}(\{s_i\}) + \sum_{\{A_i \in S_i \setminus S_i | s_i \in A_i\}} m_{\alpha_i}(A_i) \sigma_{A_i}(s_i).
\]

Furthermore, Proposition 7 in Chateauneuf and Jaffray (1989) implies that if \( \alpha_i \) is a belief function and \((\sigma_{A_i})_{A_i \in S_i \setminus S_i} \times A_i \in S_i \setminus S_i \Delta A_i \) then the function \( q_i \) defined by equation (6) is a probability measure that is also an element of \( C(\alpha_i) \).

Define, for any \( s_i \in S_i \),

\[
\Theta_i[s_i] := \{ \theta_i \in \Theta_i | \theta_i = \{s_i\} \text{ or } \theta_i \in \mathcal{F}_i \setminus \mathcal{F}_i^g \text{ and } s_i \in \theta_i \},
\]

and for any \( \theta_i \in \Theta_i \),

\[
S_i[\theta_i] := \begin{cases} 
S_i, & \text{if } \theta_i = \{s_i\}, \\
A_i, & \text{if } \theta_i = A_i \in \mathcal{F}_i \setminus \mathcal{F}_i^g.
\end{cases}
\]

\(^{14}\)Bade (2008) also introduces various behavioural notions of stochastic independence in an MEU framework, and shows them to be weaker than those proposed by Gilboa and Schmeidler (1989). Hence, since the notion of independence implicit in the definition of the SPI is equivalent to the behavioral characterization of independence of Gilboa and Schmeidler (1989), it also satisfies the independence requirements of Bade (2008).
Since $m_{a_i}(\{s_i\}) = \rho_{a_i, \pi_{a_i}}(s_i)$, we can use the notation introduced for the definition of the SPI to rewrite equation (6) as

$$q_i^{\sigma_i}(s_i) := \sum_{\theta_i \in \Theta_i[s_i]} p_{a_i}(\theta_i) \langle \pi_{a_i}^{\theta_i} \sigma_i \rangle (s_i),$$

where $q_i^{\sigma_i}$ is now defined as the element of $C(\alpha_i)$ that is induced by the particular $\sigma_i = (\sigma_{a_i})_{A_i \in \mathcal{A} \setminus \mathcal{A}_i} \in \times_{A_i \in \mathcal{A} \setminus \mathcal{A}_i} \Delta A_i = \Delta_i$ in the RHS of equation (7). The characterization of the core of a belief function $\alpha_i$ from Chateauneuf and Jaffray (1989) then implies that

$$C(\alpha_i) = \{q_i^{\sigma_i} \in \Delta S_i \mid \sigma_i \in \Delta_i\},$$

so that any $q \in \otimes_i C(\alpha_i)$ is defined across all $s \in \times_i S_i$, by

$$q(s) = \prod_i q_i^{\sigma_i}(s_i) = \prod_i \left( \sum_{\theta_i \in \Theta_i[s_i]} p_{a_i}(\theta_i) \langle \pi_{a_i}^{\theta_i} \sigma_i \rangle (s_i) \right),$$

for some $\sigma \in \times_i \Delta_i$. It follows that

$$\min_{q \in \otimes_i C(\alpha_i)} \int_{\times_i S_i} u d q = \min_{\sigma \in \times_i \Delta_i} \left\{ \sum_{s \in \times_i S_i} \left[ \prod_{\theta \in \Theta_i} \left( \sum_{\theta_i \in \Theta_i[s_i]} p_{a_i}(\theta_i) \langle \pi_{a_i}^{\theta_i} \sigma_i \rangle (s_i) \right) \right] u(s) \right\}$$

$$= \min_{\sigma \in \times_i \Delta_i} \left\{ \sum_{s \in \times_i S_i} \left[ \prod_{\theta \in \Theta_i} \left( \sum_{\theta_i \in \Theta_i[s_i]} p_{a_i}(\theta_i) \right) \left( \prod_{\theta_i} \langle \pi_{a_i}^{\theta_i} \sigma_i \rangle (s_i) \right) \right] u(s) \right\}$$

$$= \min_{\sigma \in \times_i \Delta_i} \left\{ \prod_{\theta \in \Theta_i} \left( \sum_{\theta_i \in \Theta_i[s_i]} p_{a_i}(\theta_i) \right) \left[ \sum_{s \in \times_i S_i} \left( \prod_{\theta_i} \langle \pi_{a_i}^{\theta_i} \sigma_i \rangle (s_i) \right) u(s) \right] \right\}$$

$$= \min_{\sigma \in \times_i \Delta_i} \left\{ \mathbb{E}_p \left[ \mathbb{E}_{\langle \pi^{\sigma} \sigma \rangle} [u] \right] \right\} = \int_{\times_i S_i} u d(\otimes_i \alpha_i).$$

While we have so far only considered independent products of belief functions, motivated by Proposition 1, it would be possible to extend the definition of the SPI to the case where the $\alpha_i$'s are convex capacities (in which case their cores would be guaranteed to be non-empty), by defining a product integral as $\min_{q \in \otimes_i C(\alpha_i)} \int_{\times_i S_i} u d q$. However, for the case of convex capacities that are not belief functions, the Möbius weights are not guaranteed to be non-negative, and therefore do not yield probability distributions over the type sets that motivated the definition of the SPI. As a result, not only would the type interpretation not hold anymore, but also the definition of the SPI based on nested expectations with respect to standard products of probability measures would break down. Furthermore, the equivalence result of Chateauneuf and Jaffray (1989) characterizing $C(\alpha_i)$, on which the proof of Proposition 1 is based on, only holds for belief functions but not for convex capacities. Hence, in the remainder of the paper, we maintain the restriction to belief functions.
3.4 Properties of the strategic product integral

The definition of the SPI as a double expectation preceded by a minimization operator implies that it satisfies the following common properties, which follow as a straightforward corollary of the MEU characterization from Proposition 1:

**Proposition 2.** If \( \alpha_i \) for \( i \in \{1, \ldots, n\} \) are belief functions defined on finite sets \( S_i \), then for any functions \( u, v : x_i S_i \rightarrow \mathbb{R} \),

(i) \( \hat{f}_{x_i S_i} \text{cud}(\overline{x} \alpha_i) = c \hat{f}_{x_i S_i} ud(\overline{x} \alpha_i) \) for any constant \( c \geq 0 \);

(ii) \( \hat{f}_{x_i S_i} (u + c)\overline{x} \alpha_i = \hat{f}_{x_i S_i} ud(\overline{x} \alpha_i) + c \) for any constant \( c \in \mathbb{R} \);

(iii) \( u \geq v \) implies that \( \hat{f}_{x_i S_i} ud(\overline{x} \alpha_i) \geq \hat{f}_{x_i S_i} vd(\overline{x} \alpha_i) \);

(iv) \( \hat{f}_{x_i S_i} (u + v)\overline{x} \alpha_i \geq \hat{f}_{x_i S_i} ud(\overline{x} \alpha_i) + \hat{f}_{x_i S_i} vd(\overline{x} \alpha_i) \).

The first three of these properties also hold for standard integrals and Choquet integrals, whereas the fourth property replaces an equality in the standard case, and—for a special class of functions called “co-monotonic” functions (Ghirardato, 1997)—in the Choquet case, with an inequality. Note also that one particular implication of properties (i) and (iv) is that the SPI is a concave functional.

To show that except for the one-dimensional case, the SPI cannot, in general, be expressed as the Choquet integral of any appropriately constructed capacity, we first define an independent product capacity \( \hat{\alpha} : 2^{\times_i S_i} \rightarrow [0, 1] \), by letting \( 1_A \) denote the indicator function of any set \( A \subseteq x_i S_i \), and setting

\[
\hat{\alpha}(A) := \hat{\int}_{x_i S_i} 1_A d\overline{x} \alpha_i = \min_{q \in \otimes C(\alpha_i)} \int_{x_i S_i} 1_A dq.
\]

Then \( \hat{\alpha} \) is an independent product capacity if it is a capacity, and for every \( A = \times_i A_i \subseteq \times_i S_i \),

\[
\hat{\alpha}(\times_i A_i) = \prod_i \alpha_i(A_i). \tag{8}
\]

To see that these properties are satisfied, note first that if \( A \subseteq \times_i S_i \) is empty, the indicator function \( 1_A \) is equal to zero everywhere, so \( \hat{\alpha}(A) = 0 \). Clearly, if \( A = \times_i S_i \), the indicator function is equal to 1 on its whole domain, and hence \( \hat{\alpha}(A) = \hat{\alpha}(\times_i S_i) = 1 \). Monotonicity of \( \hat{\alpha} \) then follows as a consequence of Proposition 2(iii), by noting that whenever \( A \subseteq B \subseteq \times_i S_i \), then \( 1_A \leq 1_B \). Hence, \( \hat{\alpha} \) is a capacity. To show that the product property (8) also holds, let \( A = \times_i A_i \subseteq \times_i S_i \), and consider any \( q = \times_i q_i \in \otimes C(\alpha_i) \). Then

\[
\int_{x_i S_i} 1_A dq = \int_{x_i S_i} \prod_i 1_{A_i} dq = \prod_i \int_{S_i} 1_{A_i} dq_i,
\]

and therefore, by Proposition 1,

\[
\hat{\alpha}(\times_i A_i) = \min_{q \in \otimes C(\alpha_i)} \int_{x_i S_i} 1_A dq = \prod_i \min_{q_i \in C(\alpha_i)} \int_{S_i} 1_{A_i} dq_i = \prod_i \int_{S_i} 1_{A_i} dq_i = \prod_i \alpha_i(A_i),
\]

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where the penultimate equality follows from the assumption that the $\alpha_i$ are belief functions, and hence convex capacities, and the last equality from a standard property of the Choquet integral.

Now, if $\tilde{\alpha} : 2^{\times_i S_i} \to [0, 1]$ is a capacity which satisfies that for all $u : \times_i S_i \to \mathbb{R}$,

$$\int_{\times_i S_i} u d\tilde{\alpha} = \bigwedge_{\times_i S_i} u d(\overline{\mathbb{X}}) \alpha_i, \quad (9)$$

then for every $A \subseteq \times_i S_i$ and corresponding indicator function $1_A : \times_i S_i \to \mathbb{R}$,

$$\tilde{\alpha}(A) = \int_{\times_i S_i} 1_A d\tilde{\alpha} = \bigwedge_{\times_i S_i} 1_A d(\overline{\mathbb{X}}) \alpha_i = \check{\alpha}(A),$$

so $\tilde{\alpha}$ must be identical to the product capacity $\check{\alpha}$ defined above. Since the SPI is a concave functional, it follows from Proposition 4.1 in Lovász (1983) (see also Lehrer, 2009), which shows that the Choquet integral of a capacity $\check{\alpha}$ is a concave functional if and only if $\check{\alpha}$ is convex, that the equivalence from equation (9) can only hold if $\check{\alpha}$ can be guaranteed to be a convex capacity.\(^{15}\) Appendix A shows that even for the case of the simple example considered in Section 3.1, the associated capacity $\check{\alpha}$ need not be convex.\(^{16}\) Hence, the SPI cannot be recovered by first constructing an appropriate capacity on $\times_i S_i$, and then computing its Choquet integral. We summarize this result in the following proposition:

**Proposition 3.** In general, there does not exist a capacity $\tilde{\alpha} : 2^{\times_i S_i} \to [0, 1]$ such that

$$\int_{\times_i S_i} u d\tilde{\alpha} = \bigwedge_{\times_i S_i} u d(\overline{\mathbb{X}}) \alpha_i, \text{ for all } u : \times_i S_i \to \mathbb{R}.$$

Another integral for capacities that shares the properties of the SPI in being a concave and homogeneous functional, is the “concave integral” introduced by Lehrer (2009). Hence, an interesting question is whether there is a connection between the SPI and the concave integral with respect to the product capacity $\check{\alpha}$. We refer to Lehrer (2009) for a rigorous definition and discussion of the concave integral, noting only that following Lemma 1(ii) in Lehrer (2009, p. 160), the concave integral of a non-negative function $u : S \to \mathbb{R}_+$ with respect to $\check{\alpha}$ can be expressed as

$$\int_S^{\text{cav}} u d\check{\alpha} := \min \left\{ \int_S u dP \left| P \text{ is an additive measure such that } P \geq \check{\alpha} \right. \right\},$$

where the inequality $P \geq \check{\alpha}$ is interpreted event-wise. Since $\check{\alpha}$ is the event-wise lower envelope of the product set $\otimes_i C(\alpha_i)$, it follows that the minimum in the definition of $\int_S^{\text{cav}} u d\check{\alpha}$ will in general be taken over a set that is larger than $\otimes_i C(\alpha_i)$, as it also contains measures that allow for correlation (as pointed out by Ghirardato, 1997, p. 282). The concave integral of a function $u$ with respect to $\check{\alpha}$ will, however, coincide with the respective SPI as long as the minimum in the MEU characterization of

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\(^{15}\)I thank an anonymous referee for pointing out this connection.

\(^{16}\)Appendix A also shows that $\check{\alpha}$ need not be a belief function either, and therefore implies that in general, $\check{\alpha}$ cannot be equal to the Möbius product of the $\alpha_i$, as the Möbius product of belief functions must always be a belief function (Ghirardato, 1997).
the SPI is equal to the minimum in the definition of the concave integral. This consideration then yields the following sufficient condition for equality of the two integrals (compare also Theorem 2 in Lehrer, 2009, p. 164):

**Proposition 4.** Let \( u : S \to \mathbb{R}_+ \) be a non-negative function. Then the following two statements are equivalent:

(i) \( \int_S \text{cav} \, ud\hat{\alpha} = \int_S ud(\overline{x}_i a_i) \);

(ii) \( \int_S udP \geq \min_{\gamma \in C(\hat{\alpha})} \int_S ud\gamma \), for all additive measures \( P \) satisfying \( P \geq \hat{\alpha} \).

4 Normal-form games and equilibrium in belief functions

4.1 Preferences for randomization

The equilibrium notion we propose allows players to explicitly randomize by choosing their actions based on the outcome of an independent randomization device. To motivate this modeling choice, and to clarify the interaction between such randomizations and ambiguity-sensitive preferences, this subsection provides a brief discussion of the literature on preferences for randomization with ambiguity aversion, and relates it to corresponding properties of the SPI.

As noted by Raiffa (1961), in the presence of ambiguity, agents may exhibit a strict preference for (independent) randomization over their action choices. Following Raiffa (1961), an extensive literature has developed that characterizes such preferences in the context of various preference representations. In these settings, the randomizations are essentially assumed to be objective, in the sense that the players are able to commit to delegate their actions to an independent randomization device. For MEU preferences, Klibanoff (1996) and Lo (1996) show that players in a normal-form game with ambiguous beliefs regarding the strategies of their opponents, may strictly prefer a mixed strategy over any of their pure actions.\(^{17}\) Eichberger and Kelsey (1996) discuss preferences for randomization in a CEU setting, and show that strict preferences for randomization may or may not arise depending on the decision-theoretic representation used to define expected payoffs. Specifically, they show that if an Anscombe-Aumann framework is used where the randomization is built into the objective lotteries component of the utility representation, an agent may exhibit strict preferences for randomization, however, if the randomization is included directly in the state space of a Savage-style framework, no strict preferences for randomization can arise if the beliefs over the underlying uncertainty are represented by a convex capacity. Ghirardato (1997) demonstrates how this result follows from the fact that the Fubini property of iterated integrals does not carry over to product capacities, so the Anscombe-Aumann framework, which relies on a particular type of iterated

\(^{17}\)See also Riedel and Sass (2014), Aryal and Stauber (2014) and Stauber (2017) for related game-theoretic settings with MEU preferences where mixed strategies may be strictly optimal.
expectation, is not equivalent to the Choquet integral of a product over the underlying uncertainty captured by a given capacity, and the independent randomization described by a probability measure. Klibanoff (2001) provides yet another perspective on the result of Eichberger and Kelsey (1996), by introducing a behavioral, preference-based condition for stochastic independence of a randomization device, and showing that if this condition holds, CEU preferences in a Savage framework which satisfy uncertainty aversion, in the sense that they are represented by a convex capacity, must collapse to expected utility. The conclusion he draws from this result is that CEU preferences are not appropriate to simultaneously model both uncertainty aversion and an independent randomization device.

Appendix B illustrates the results discussed above using a classic example from Raiffa (1961), also analyzed in Eichberger and Kelsey (1996) and Klibanoff (2001), and shows how the SPI can be used to reconcile the various conclusions. Accordingly, a strict preference for randomization reappears when the SPI is used to compute the expected utility over a product space involving an independent probabilistic randomization and a given underlying uncertainty, even if acts are defined using a Savage framework. This result is a consequence of the MEU characterization of the SPI from Proposition 1, which allows a direct application of Theorem 1 from Klibanoff (2001) to confirm conditions under which independent randomizations are strictly beneficial. Hence, even though CEU cannot be used to model independent randomization if the expected utility of the product is computed based on a Choquet integral of a product capacity, such independent randomizations may still be modeled in conjunction with ambiguous beliefs represented by capacities, as long as these are restricted to belief functions, and the corresponding product integral is defined by the SPI. This result provides a compelling motivation for using the SPI instead of the Möbius product to model independent products, since, even if one is not willing to exclude the possibility of correlation between opponents’ actions in the presence of ambiguity, such correlations should not be feasible for a randomization device that is deliberately chosen to be stochastically independent from opponents’ strategies.

4.2 Equilibrium definition and existence

We now apply the SPI to introduce a notion of equilibrium for finite, simultaneous-action, normal-form games. As discussed in the previous subsection, if a decision-maker’s beliefs regarding some underlying uncertainty can be represented by a belief function, and expected payoffs for independent products are defined based on the SPI, such a decision-maker may strictly prefer to delegate his actions to an objective randomization device that is independent of the uncertainty he faces. In a normal-form game, this implies that a player who faces uncertainty regarding the action choices of his opponents, may have a strict preference to choose a mixed strategy that hedges his payoffs against this uncertainty.
Computing the players’ expected payoffs based on the SPI is equivalent to imposing an independence assumption across all players’ action choices, including an assumption that each player’s individual mixed strategy is chosen independently from his opponents’ strategies. We view this independence assumption, which could follow from either a physical restriction or an epistemic requirement, as a benchmark case, analogous to the independence requirement of Nash equilibrium. Using the Möbius product instead of the SPI would implicitly allow for correlations between the actions of various strategic types. While constructing a theory that allows for such correlations would in principle be possible, we believe that if the strategic environment does allow for correlations, there is no reason to only exclude correlations between rational types, so correlations between all types should be allowed. A resulting general theory of correlated equilibrium under strategic uncertainty would yield an interesting question for future research, but given that NE is still much more widely used in applications, there seems to be merit in first addressing the case with independent actions.

We consider finite normal-form games defined by

- a finite number \( N \) of players, with individual players \( i, j \in \{1, \ldots, N\} =: \mathcal{N} \),
- for each player \( i \in \mathcal{N} \), a finite action set \( S_i \), and
- for each player \( i \in \mathcal{N} \), a payoff/utility function \( u_i : \times_{j \neq i} S_j \to \mathbb{R} \).

An equilibrium will then be defined by a profile of belief functions \( \alpha_i : 2^{S_i} \to \mathbb{R} \), for \( i \in \mathcal{N} \), where, in contrast to the equilibrium under ambiguity literature, the \( \alpha_i \) does not describe player \( i \)'s beliefs. Instead, \( \alpha_i \) incorporates both the actual mixed strategy chosen by player \( i \), through the conditional distribution over singleton subsets of \( S_i \) induced by its Möbius transform, as well as the ambiguous beliefs of the opponents of \( i \) regarding \( i \)'s strategies, as captured by \( \alpha_i \) itself. This approach allows the introduction of strategic uncertainty, while maintaining a precise notion of consistency between actual probabilistic mixed strategies and ambiguous beliefs about such strategies. Consequently, player \( i \)'s mixed strategy is required to be optimal given \( i \)'s beliefs about his opponents’ strategies, as represented by all \( \alpha_j \) with \( j \neq i \). Thus, our equilibrium definition considers any individual player who himself is aware of his own rationality and does not face any strategic uncertainty regarding his own action choices, and must decide on the choice of a mixed strategy that maximizes his expected utility as computed by an SPI, given that his beliefs regarding the strategies of his opponents are given by the belief functions \( \alpha_j \). The Möbius weights attached to non-singleton sets of opponents’ actions are then assumed to capture the player’s beliefs that these opponents do not behave according to the equilibrium optimality condition. These weights allow for a degree of ambiguity that is still consistent with equilibrium, but takes into account the possibility that any opponent \( j \) may be of an alternative strategic type, who is only known to choose an action from a non-singleton subset of actions \( A_j \subseteq S_j \) according to some arbitrary procedure.
If \( \alpha_i \) specifies the equilibrium beliefs of player \( i \)'s opponents, and \( \rho_{\alpha_i} \) and \( \pi_{\alpha_i} \) are defined based on \( \alpha_i \) as in Section 3.2, we interpret \( \rho_{\alpha_i} \) as the probability assigned by \( i \)'s opponents to a rational type of \( i \), and \( \pi_{\alpha_i} \) as the mixed strategy chosen by this type. If \( \rho_{\alpha_i} = 1 \), \( \alpha_i \) is an additive measure, and \( i \)'s opponents believe that he plays a regular mixed strategy defined by a probability measure over \( S_i \). However, if \( \rho_{\alpha_i} < 1 \), the opponents face strategic uncertainty regarding \( i \)'s strategies, and there exists at least one non-singleton subset \( A_i \) of \( S_i \) such that \( m_{\alpha_i}(A_i) > 0 \). For such an \( A_i \subseteq S_i \), we interpret \( m_{\alpha_i}(A_i) \) as the probability that the opponents of \( i \) assign to the strategic type \( A_i \) of player \( i \), who is only known to play some action in \( A_i \) according to an unknown distribution from \( \Delta A_i \). This interpretation is consistent with the Dempster-Shafer theory of evidence, as the weights \( m_{\alpha_i}(A_i) \) could be viewed as reflecting objective information supporting different theories of how various players might approach playing the game.\(^\text{18}\)

From the point of view of a rational type of \( i \), his expected utility depends on the chosen mixed strategy \( \pi_i \), and his beliefs regarding the strategies of each of his opponents \( j \neq i \), as represented by the belief functions \( \alpha_j \). By viewing \( \pi_i \) as a belief function that only assigns strictly positive Möbius weight to singleton subsets of \( S_i \), \( i \)'s expected utility can be defined as the SPI of \( u_i \) computed with respect to

\[
\pi_i \boxtimes \alpha_{-i} := \alpha_1 \boxtimes \cdots \alpha_{i-1} \boxtimes \pi_i \boxtimes \alpha_{i+1} \boxtimes \cdots \alpha_N. 
\]

Hence, letting

\[
U_i(\pi_i, \alpha_{-i}) := \int_{X_i \times S_i} u_i d(\pi_i \boxtimes \alpha_{-i}), 
\]

we get the following equilibrium definition:

**Definition 1.** An equilibrium in belief functions (EBF) of a normal-form game is defined by a profile \((\alpha_1, \ldots, \alpha_N)\) of belief functions, such that for every \( i \in \mathcal{N}, \rho_{\alpha_i} > 0 \), and

\[
\pi_{\alpha_i} \in \arg \max_{\pi \in \Delta S_i} U_i(\pi_i, \alpha_{-i}).
\]

Any equilibrium belief function \( \alpha_i \) defines a unique corresponding mixed strategy \( \pi_{\alpha_i} \) (since \( \rho_{\alpha_i} > 0 \)), and a unique corresponding distribution \( p_{\alpha_i} \) over strategic types in \( \Theta_i := \mathcal{S}_i \cup (\mathcal{S}_i \setminus \mathcal{S}_i^\emptyset) \). While the mixed strategy \( \pi_{\alpha_i} \) is required to satisfy the equilibrium optimality conditions, no particular restrictions are imposed on \( p_{\alpha_i} \). Hence, one approach to deriving an EBF would be to consider an exogenously specified profile of distributions over strategic types \( p_i \in \Delta \Theta_i \), with \( p_i(\mathcal{S}_i^\emptyset) = \rho_i > 0 \), and finding a profile of mixed strategies \( \pi_i \in \Delta S_i \) that satisfy the optimality conditions in Definition 1 for the unique belief functions \( \alpha_i(p_i, \pi_i) \) that are defined by the Möbius coefficients constructed from \( p_i \) and \( \pi_i \) using the reverse of the procedure used to derive \( p_{\alpha_i} \) and \( \pi_{\alpha_i} \) from \( m_{\alpha_i} \). For example, for

\(^{18}\)As discussed in Marinacci (2000) in the context of a different modeling setup, such information could include factors that are not explicitly modeled in the game, such as past experience, preplay communication or cultural norms.
the extensive games considered in Stauber (2017), \( \rho_i = 1 - \varepsilon \), where \( \varepsilon \in (0, 1) \) is a small number, and \( p_i(S_i) = \varepsilon \), so there are only two strategic types, a rational type, and the irrational type who chooses an action from \( S_i \) according to an arbitrary distribution.\(^{19}\) For any mixed strategy \( \pi_i \) of \( i \), the induced belief function \( \alpha_{(p_i, \pi_i)} \) is then the simple capacity with \( \alpha_{(p_i, \pi_i)}(A_i) = (1 - \varepsilon)\pi_i(A_i) \) for \( A_i \subseteq S_i \).

Using this approach of deriving belief functions \( \alpha_{(p_i, \pi_i)} \) from \( p_i \)'s and \( \pi_i \)'s, it can easily be seen that every Nash equilibrium (NE) \((\pi_i^*)\) constitutes an EBF, by setting \( p_i(\{}S_i^*\}) = 1 \). This also implies that, technically, an EBF exists for any finite game. Since the optimality conditions for an EBF can, however, be different from those for NE when \( p_{a_i}(\{}S_i^*\}) < 1 \), the mixed strategy profile \((\pi_{a_i})_i \) corresponding to an EBF \((\alpha_i)_i \) need not be a NE, as we will see later in the context of an example.

If we consider any exogenously fixed profile \((p_i)_{i \in \mathcal{N}} \) with \( p_i(\{}S_i^*\}) > 0 \), and any \( \pi_{-i} \in \times_{j \neq i} S_j \), and define
\[
\alpha_{-(p_i, \pi_i)} := (\alpha_{(p_1, \pi_1)}, \ldots, \alpha_{(p_{i-1}, \pi_{i-1})}, \alpha_{(p_{i+1}, \pi_{i+1})}, \ldots, \alpha_{(p_N, \pi_N)}),
\]
then the definition of the SPI implies that \( U_i(\pi_i, \alpha_{-(p_i, \pi_i)}) \) is continuous and concave in \( \pi_i \). A standard application of Kakutani’s fixed point theorem then yields the following existence result:

**Proposition 5.** For any given profile of distributions over rationality types \((p_i)_{i \in \mathcal{N}} \) with \( p_i(\{}S_i^*\}) = \rho_i > 0 \), there exists a profile of mixed strategies \((\pi_i)_{i \in \mathcal{N}} \), such that \((\alpha_{(p_i, \pi_i)})_{i \in \mathcal{N}} \) constitutes an EBF.

**Proof.** Using the MEU representation of the SPI, the best-response correspondence of any player \( i \) can be expressed as
\[
BR_i(\pi_{-i}) := \arg\max_{\pi_i \in \Delta S_i} U_i(\pi_i, \alpha_{-(p_i, \pi_i)}) = \arg\max_{\pi_i \in \Delta S_i} \left\{ \min_{q_{-i} \in \mathcal{R}_{j \neq i}} \left[ \min_{1 \leq j \leq \mathcal{N}} C_j(\alpha_{(p_i, \pi_i)}) \right] \right\}.
\]
The characterization of the core of a belief function from Chateauneuf and Jaffray (1989) discussed in the proof of Proposition 1, implies that
\[
C_j(\alpha_{(p_i, \pi_i)}) = \rho_j \pi_j + \sum_{A_j \in \mathcal{R}_j \setminus S_j} p_j(A_j) \Delta A_j,
\]
which is continuous in \( \pi_j \) (using the Hausdorff metric, for example). Hence, \( U_i(\pi_i, \alpha_{-(p_i, \pi_i)}) \) is continuous in \((\pi_i, \pi_{-i})\), and concave in \( \pi_j \) for every \( \pi_{-j} \), since the integral \( \int u_i(q_{-i}) \mathrm{d} \pi_{-i} \) is linear in \( \pi_i \) for every \( q_{-i} \). As \( \Delta S_i \) is constant, and therefore a continuous and compact-valued correspondence in \( \pi_{-i} \), it follows from the Maximum Theorem that \( BR_i \) is non-empty, upper-hemicontinuous and compact- and convex-valued. A direct application of Kakutani’s fixed point theorem to the correspondence \( \times_{i \in \mathcal{N}} BR_i \) then yields the existence of an equilibrium. \( \Box \)

\(^{19}\)A similar framework is also used by Renou and Schlag (2010), who define a notion of \( \varepsilon \)-minimax regret equilibrium, in which each player believes with probability \( 1 - \varepsilon \) that his opponents are playing according to an equilibrium strategy, and is completely uncertain regarding the play of his opponents with complementary probability \( \varepsilon \); in contrast to our setting, players are however assumed to use a decision rule that minimizes the expected maximum regret over the potential strategies of opponents.
4.3 Properties and discussion

As commonly recognized in the literature on games with strategic uncertainty (Klibanoff, 1996; Marinacci, 2000; Eichberger and Kelsey, 2000), in situations of extreme ambiguity, where no information regarding opponents’ strategies are available, only strategy choices that achieve a maxmin over opponents’ action spaces are optimal. To model such cases of extreme ambiguity using EBF, we need to remove the assumption that \( \rho_{\alpha_i} > 0 \). If we thus set \( \rho_{\alpha_i} = 0 \) for every \( i \), a mixed strategy \( \pi_{\alpha_i} \) is not uniquely associated with \( \alpha_i \). Hence, if we want to maintain the interpretation of \( \alpha_i \) as representing the beliefs of \( i \)'s opponents regarding \( i \)'s strategy, it is necessary to specify the mixed strategy of a rational type of \( i \), \( \pi_{\alpha_i} \), separately from \( \alpha_i \). An equilibrium can then be defined by removing the restriction \( \rho_{\alpha_i} > 0 \) from Definition 1, and adding to the equilibrium specification a profile of mixed strategies \( (\pi^*_i)_{i \in \mathcal{N}} \) that satisfy

\[
\pi^*_i \in \arg \max_{\pi_i \in \Delta S_i} U_i(\pi_i, \pi_{-i}).
\]

Clearly, when \( \rho_{\alpha_i} = 0 \), the equilibrium strategy \( \pi^*_i \) of the rational type of player \( i \) has no impact on his opponents’ expected payoffs. Even when \( \rho_{\alpha_i} = 0 \), defining the beliefs of \( i \)'s opponents regarding the strategy of \( i \) using a belief function \( \alpha_i \) still allows variability regarding the remaining ambiguity, as captured by the weights \( m_{\alpha_i}(A_i) \equiv p_i(A_i) \) assigned to non-singleton sets \( A_i \subseteq S_i \). A situation with extreme ambiguity could then be described by belief functions \( \alpha_i \) for which \( m_{\alpha_i}(S_i) = 1 \). An alternative situation where the rational type of \( i \) is still assigned zero probability, but where the opponents have sufficient information to restrict \( i \)'s actions to a non-singleton subset \( A_i \subseteq S_i \), could be described by a belief function with \( m_{\alpha_i}(A_i) = 1 \), so our framework is able to describe more limited cases of extreme ambiguity. The EBF corresponding to such situations are then characterized by the following proposition (the proof follows directly from the definitions and is omitted):

**Proposition 6.** Let \( (\alpha_i)_{i \in \mathcal{N}} \) be a profile of belief functions such that \( \rho_{\alpha_i} = 0 \), and \( m_{\alpha_i}(A_i) = 1 \) for some non-singleton set \( A_i \subseteq S_i \). Then \( (\pi^*_i, \alpha_i)_{i \in \mathcal{N}} \) constitutes an EBF if and only if

\[
\pi^*_i \in \arg \max_{\pi_i \in \Delta S_i} \min_{s_{-i} \in \times_{j \neq i} A_j} U_i(\pi_i, s_{-i}) \text{ for all } i \in \mathcal{N}.
\]

An opposite extreme case arises when the degree of ambiguity converges to zero, so that \( \rho_{\alpha_i} \rightarrow 1 \) for all \( i \). Consider, for example, a sequence of EBF \( (\alpha^n_i)_i \) such that \( \rho_{\alpha^n_i} \rightarrow 1 \), and \( (\alpha^n_i)_i \) is convergent, in the sense that the probability distributions over \( \mathcal{S}_i \) induced by the Möbius transforms of the \( \alpha^n_i \) converge. Analogous to the definition of a trembling-hand-perfect equilibrium (THPE) (Selten, 1975), we can then define a limit EBF as the limit \( \bar{\pi}_i := \lim_{n \rightarrow \infty} \pi_{\alpha^n_i} \) of the equilibrium mixed strategies corresponding to each \( \alpha^n_i \) (which must converge since each \( \alpha^n_i \) was assumed to be convergent). As \( \rho_{\alpha^n_i} \rightarrow 1 \) and \( \pi_{\alpha^n_i} \rightarrow \bar{\pi}_i \), it follows that \( C_i(\alpha^n_i) \rightarrow \bar{\pi}_i \) (see equation (10)), and therefore the optimality of \( \pi_{\alpha^n_i} \) for each \( n \) implies that
Proposition 7. For any convergent sequence \( (\alpha^n_i) \) that satisfies \( \rho_{\alpha^n_i} \to 1 \), the associated limit EBF \( (\pi_i)_{i \in \mathcal{N}} \) must be a NE.

Hence, the notion of limit EBF can be used to characterize those NE that are robust to the introduction of a small amount of ambiguity.\(^{20}\) The resulting refinement is not equivalent to THPE, as the EBF strategies \( \pi_{\alpha^n} \) need not necessarily only be best responses to completely mixed strategies of the opponents, as required by THPE. This can be seen in the game described in Figure 1, where BR is an EBF if the players’ ambiguous beliefs assign a small positive probability \( \varepsilon > 0 \) to the respective irrational types \( \{T, B\} \) and \( \{L, R\} \), even though B and R are weakly dominated (this follows from the fact that the worst-case action for player 1 is 2 playing R, and the worst-case action for player 2 is 1 playing B, irrespective of the players’ own strategies). There do, however, also exist games in which ambiguity eliminates weakly dominated strategies as part of an EBF, and therefore as part of a limit EBF—for example, in the game described in Figure 2, which is drawn from Eichberger and Kelsey (2000), B and R are weakly dominated, and cannot be played in any EBF if the players’ ambiguous beliefs assign probability \( \varepsilon > 0 \) to the types \( \{T, B\} \) and \( \{L, R\} \). A further three-player game that illustrates the fact that limit EBF can yield a non-trivial refinement of NE, even when no action is weakly dominated for any player, is presented as Example 3 in Appendix C. An additional difference between limit EBF and THPR, is that, even with consistency of ambiguous beliefs across players, two players may best-respond to different worst-case strategies of a third opponent. Overall, while the resulting notion of robustness to ambiguity provides an intuitive method for selecting among NE, its relevance as a refinement seems to be somewhat limited in the context of normal-form games. However, using a more basic framework restricted to simple capacities, Stauber (2017) shows that an analogous approach to selecting NE that are robust to ambiguity in the context of an extensive game, can have strong implications, so that in particular, subgame-perfect equilibria may not be robust to any small amount of ambiguity.

We conclude this subsection with a discussion of various properties of EBF, and relate them to alternative notions of equilibrium with ambiguous beliefs that have been analyzed in the literature:

- Modeling beliefs using belief functions that are characterized by their Möbius transforms, al-

\(^{20}\) A similar result is also presented in Eichberger and Kelsey (2000), however, since their equilibrium concept does not allow for mixed strategies and does not require independence, their result and proof require various additional considerations and assumptions.
lows EBF to be defined through a type of semi-consistency, which precisely embeds standard mixed strategies as objects of choice, into ambiguous beliefs about strategies that are described by belief functions. This is in contrast to the equilibrium notions of Klibanoff (1996) and Lo (1996), for which there is no intrinsic connection between equilibrium strategies and beliefs. As discussed in the introduction, the consistency notion defining EBF is closer in spirit to the partially specified equilibrium of Lehrer (2012), although the two differ in the way the equilibrium mixed strategies are embedded into equilibrium beliefs—in EBF, this embedding is achieved through the Möbius transform, whereas in partially specified equilibrium it is achieved by limiting a player’s opponents’ information regarding the player’s equilibrium mixed strategy.

- Using the SPI to capture independence of a player’s own mixed strategy to beliefs about his opponents’ strategies, and across the strategies of opponents, recovers the possibility of strict preferences for independent randomization, which cannot arise when payoffs are computed using the Choquet integral. This provides an additional hedging motive for randomization, in which case mixed strategies that are part of an EBF may be strictly preferred to any alternative strategies.\textsuperscript{21} Furthermore, the SPI provides an intuitive approach for defining expected payoffs with independent strategies and beliefs, which is behaviorally equivalent to the well-known Gilboa-Schmeidler independence notion for MEU preferences.\textsuperscript{22}

- Interpreting equilibrium mixed strategies as explicit randomizations, as opposed to “equilibria in beliefs,” as in Dow and Werlang (1994) or Eichberger and Kelsey (2000), allows equilibrium strategies to be viewed as characterizing the likelihood of various outcomes of a game. As a consequence of this interpretation, EBF can be used to analyze how such outcomes vary as a function of changes in the nature of the ambiguity that describes the strategic situation, and how the interaction between the degree of ambiguity and other parameters describing the game may affect the resulting equilibrium outcomes.

\textsuperscript{21}As the examples analyzed in Appendix C show, not all EBF in mixed strategies yield such strict equilibria. Conditions under which strict preferences for randomization can arise in game-theoretic settings are derived in Klibanoff (1996).

\textsuperscript{22}Note that the partially specified equilibrium of Lehrer (2012) also interprets mixed strategies as objective randomizations, and assumes an independence requirement analogous to Gilboa-Schmeidler independence.
4.4 Examples

In Appendix C, we present four examples that illustrate various properties of EBF. This subsection summarizes the principal insights arising from these examples.

Example 1 considers a simple matching pennies game, in which the presence of ambiguity yields a strict preference for randomization. When ambiguity levels are sufficiently small, the unique resulting EBF involves equilibrium strategies where the two actions are played with equal probabilities, which seems equivalent to the usual outcome under expected utility. However, in the presence of ambiguity, hedging motives imply that these equilibrium strategies are strictly preferred to any alternative pure or mixed strategies, even for very small ambiguity levels. Hence, in contrast to the standard analysis under expected utility, the introduction of strategic uncertainty can provide a strict incentive to play mixed strategies, and in this sense, a stronger justification for the predictions of the standard model.

Example 2 illustrates how the nature of the ambiguity that is assumed can result in distinct equilibrium predictions, and shows how the probability distributions over strategic types that arise from the Möbius transforms can be used in an effective way to specify different requirements on the nature of the assumed ambiguity. In particular, the example compares equilibrium predictions for the case where the strategic uncertainty is restricted to undominated actions, to the case where any arbitrary action can be played by an irrational type.

The principal insight arising from Example 3 is that not all NE of a game without ambiguity will be robust to the introduction of a small amount of ambiguity. Using a three-player game with no dominated actions, which also illustrates how the SPI can be used to define expected payoffs with individual mixed strategies and ambiguity regarding multiple opponents’ strategies, it shows that the set of limit EBF can be a strict subset of the set of NE.

Example 4 analyzes a simple three-player costly voting game, and shows how the interaction between voting cost and degree of ambiguity determines the resulting equilibrium outcomes. The interpretation of mixed strategies defining an EBF as explicit randomizations allows the characterization of equilibrium outcomes in terms of the associated voting probabilities and expected voting costs. In particular, the equilibrium voting probabilities decrease as a result of an increase in the degree of ambiguity. Furthermore, an equilibrium outcome that yields the socially efficient voting outcome at the lowest total expected voting cost is shown to exist only when the degree of ambiguity is strictly positive and satisfies a specific condition relative to the voting cost.
5 Concluding remarks

The presented definition of EBF can be viewed as a baseline model, which assumes beliefs that are consistent across players, as well as no correlations between strategic types or strategies. The definition of the SPI as a double expectation preceded by a minimization operator, does, however, also provide a convenient departure point for a number of potential extensions. EBF assumes that all players’ beliefs regarding the distribution of strategic types of their opponents are consistent, in the sense that the same \( p_{\alpha_i} \) describes the beliefs of all opponents of \( i \) regarding the likelihood of the various types of \( i \). One possible way to motivate such an assumption would be along the lines of the Dempster-Shafer theory of evidence, by viewing consistency as a consequence of all players’ basing their beliefs over strategic types on the same publicly available evidence or information. The definition of EBF could easily be generalized by allowing inconsistent beliefs over strategic types, so that each player \( j \) has individual beliefs \( p^j_{\alpha_i} \in \Delta \Theta_i \) regarding the strategic types of \( i \), and only requiring consistency of beliefs for the equilibrium mixed strategies of the rational types of players. In such a model, expected payoffs could still be computed using the SPI, and equilibrium existence would follow as before. One question that would arise for this extended setting, would be whether such differences in beliefs should then be viewed as common knowledge, or whether it would be more natural to consider a further extension, where players can have incomplete information regarding their opponents’ beliefs over strategic types.

An alternative extension of our model could instead allow for correlations in the joint distribution over strategic types. Hence, instead of computing the SPI based on an independent product measure over strategic types, the game could be defined by a prior distribution over strategic types, and each rational type could compute his expected payoff in the game using the conditional of this prior, given his type, for the outer expectation in the definition of the SPI. The definitions of SPI and EBF could easily be adjusted to allow for such correlations in strategic types, however, the maxmin characterization of the SPI would not hold for this generalized setting. Note also that such correlations between types are distinct from correlations between strategies, conditional on a type profile, the introduction of which would lead to a theory of correlated equilibrium under strategic uncertainty.

The fact that EBF is based on preferences represented by belief functions implies that players are implicitly assumed to be ambiguity averse, and hence take a pessimistic view regarding the ambiguous components of \( \alpha_{-i} \). However, following Marinacci (2000), a utility representation for an optimistic player could be defined by

\[
\bar{U}_i(\pi_i, \alpha_{-i}) := - \int_{\pi_i, \alpha_{-i}} -u_i d(\pi_i \boxtimes \alpha_{-i}),
\]

or a representation that combines various attitudes towards optimism and pessimism could be con-
structured by considering a convex combination

$$\lambda U_i(\pi_i, \alpha_i) + (1 - \lambda) \Pi_i(\pi_i, \alpha_i),$$

with $\lambda \in [0, 1]$, similar to Eichberger and Kelsey (2014) and Dominiak and Eichberger (2017), who consider analogous payoffs in a CEU framework. The definition of EBF could then easily be generalized to games where any player’s degree of pessimism or optimism is parametrized by $\lambda$. Replacing $\pi_i$ with $\alpha_i$ in the expression from equation (11), would also yield a generalization of the SPI for simple capacities, to an analogous product integral for the "neo-additive" capacities axiomatized by Chateauneuf, Eichberger and Grant (2007), which extend the Ellsberg interpretation of simple capacities by providing a parametric representation for preferences (where the respective parameter corresponds to $\lambda$ in our notation) that accommodates both pessimistic and optimistic attitudes towards uncertainty.

A further extension of the concepts introduced in this paper would involve an application to extensive games. The key questions that would need to be answered by such an extension are how to specify appropriate strategic types for extensive games, and how to define a suitable updating method in such a context. While a full development would be too expansive to include in the present paper, we note that Stauber (2017) essentially provides an application to extensive games of an SPI for simple capacities, introducing an equilibrium notion for such games that is shown to avoid the counterfactual reasoning associated with backward induction. Stauber (2017) considers extensive games where each player may be irrational with a small probability $\varepsilon$, in which case no predictions can be made about the player’s strategy, which is modeled as the set of all randomizations over the actions available at each information set. Beliefs about opponents’ rationality are then derived at each information set using full Bayesian updating, and an equilibrium is defined using the consistent planning approach (Strotz, 1955-1956; Siniscalchi, 2011). The resulting limit equilibria as the degree of ambiguity $\varepsilon$ converges to zero always constitute Nash equilibria, but are not necessarily subgame perfect. This follows from the presence of irrational types, which leads to the unraveling of the usual arguments supporting subgame perfection.

**Appendix A**  An example showing that $\hat{\alpha}$ need not be convex

Consider again the example from Section 3.1, with $S_1 = \{a, b\}$ and $S_2 = \{x, y\}$, and beliefs over $S_1$ and $S_2$ represented by simple capacities $\alpha_1$ and $\alpha_2$, with additive parts $\pi_1$ and $\pi_2$, and degrees of ambiguity $\delta_1$ and $\delta_2$, respectively. Then, as shown in Section 3.4, the set function $\hat{\alpha} : 2^{S_1} \times S_2 \to \mathbb{R}$ defined by

$$\hat{\alpha}(A) = \int_{S_1} \int_{S_2} 1_A d(\alpha_1 \boxplus \alpha_2),$$

29
is an independent product capacity of \( a_1 \) and \( a_2 \). This appendix analyzes the capacity \( \hat{\alpha} \) and its Möbius transform \( m_\hat{\alpha} \), to show that \( \hat{\alpha} \) is neither a belief function nor a convex capacity.

Consider first a singleton set such as \( A_1 \times A_2 = \{(a, x)\} \). Then it is easy to see that

\[
\hat{\alpha}(\{(a, x)\}) = (1 - \delta_1)(1 - \delta_2)\pi_1(a)\pi_2(x),
\]

since \( E_{\pi_1 \times \pi_2}[\mathbb{1}_{\{(a, x)\}}] = \pi_1(a)\sigma_2(x) \), \( E_{\pi_1 \times \pi_2}[\mathbb{1}_{\{(a, x)\}}] = \sigma_1(a)\pi_2(x) \), and \( E_{\pi_1 \times \pi_2}[\mathbb{1}_{\{(a, x)\}}] = \sigma_1(a)\sigma_2(x) \), so the minimizing values of \( \sigma_1 \) and \( \sigma_2 \) for \( u = \mathbb{1}_{\{(a, x)\}} \) in equation (3) are \( \sigma_1(a) = 0 \) and \( \sigma_2(x) = 0 \). The value of \( \hat{\alpha} \) for all other singletons can be derived analogously. If one of \( A_1 \) or \( A_2 \) is a singleton, and the other the full space \( S_1 \) or \( S_2 \), such as, for example, when \( A_1 = \{a\} \) and \( A_2 = \{x, y\} \), we can substitute \( u = \mathbb{1}_{\{(a, x), (a, y)\}} \) in equation (3), which results in an expression that is independent of \( \sigma_2 \), and since the minimizing value of \( \sigma_1 \) satisfies \( \sigma_1(a) = 0 \), this implies that

\[
\hat{\alpha}(\{(a, x), (a, y)\}) = (1 - \delta_1)(1 - \delta_2)\pi_1(a) + (1 - \delta_1)\delta_2\pi_1(a) = (1 - \delta_1)\pi_1(a).
\]

Claim 1. \( \hat{\alpha} \) is not a belief function.

Proof. Following Gilboa and Schmeidler (1994, Theorem 3.4), the capacity \( \hat{\alpha} \) is a belief function if and only if its Möbius coefficients \( m_\hat{\alpha}(A) \) are all non-negative. We will show that this cannot hold for \( \hat{\alpha} \).

Since \( \hat{\alpha}(A) = \sum_{B \subseteq A} m_\hat{\alpha}(B) \), it follows that

\[
\hat{\alpha}(\{(a, x)\}) = m_\hat{\alpha}(\{(a, x)\}) = (1 - \delta_1)(1 - \delta_2)\pi_1(a)\pi_2(x),
\]

and analogously for all other singleton sets. For \( A = \{(a, x), (a, y)\} \), we thus have

\[
\hat{\alpha}(\{(a, x), (a, y)\}) = (1 - \delta_1)\pi_1(a) + m_\hat{\alpha}(\{(a, x)\}) + m_\hat{\alpha}(\{(a, y)\}) + m_\hat{\alpha}(\{(a, x), (a, y)\}),
\]

which implies that

\[
m_\hat{\alpha}(\{(a, x), (a, y)\}) = (1 - \delta_1)\pi_1(a) - m_\hat{\alpha}(\{(a, x)\}) - m_\hat{\alpha}(\{(a, y)\}) = (1 - \delta_1)\delta_2\pi_1(a).
\]

Consequently, for the sets \( \{(a, x), (b, y)\} \) and \( \{(a, x), (a, y), (b, y)\} \), we get

\[
\hat{\alpha}(\{(a, x), (b, y)\}) = (1 - \delta_1)(1 - \delta_2)\pi_1(a)\pi_2(x) + (1 - \delta_1)(1 - \delta_2)\pi_1(b)\pi_2(y) + m_\hat{\alpha}(\{(a, x), (b, y)\}),
\]

and

\[
\hat{\alpha}(\{(a, x), (a, y), (b, y)\}) = (1 - \delta_1)(1 - \delta_2)[\pi_1(a) + \pi_2(b)\pi_2(y)] + (1 - \delta_1)\delta_2\pi_1(a) + \delta_1(1 - \delta_2)\pi_2(y) + m_\hat{\alpha}(\{(a, x), (a, y), (b, y)\}) + m_\hat{\alpha}(\{(a, x), (a, y), (b, y)\}).
\]
Using the notation $s_1 := \sigma_1(a)$ and $s_2 := \sigma_2(x)$, the definition of $\hat{\alpha}$ also implies that
\[
\hat{\alpha}((a, x), (a, y), (b, y)) = (1 - \delta_1)(1 - \delta_2)[\pi_1(a) + \pi_1(b)\pi_2(y)] + \min_{(s_1, s_2) \in [0,1]^2} \left\{ (1 - \delta_1)\delta_2[\pi_1(a) + \pi_1(b)(1 - s_2)] + \delta_1(1 - \delta_2)[s_1 + (1 - s_1)\pi_2(y)] + \delta_1\delta_2[s_1 + (1 - s_1)(1 - s_2)] \right\} = (1 - \delta_1)(1 - \delta_2)[\pi_1(a) + \pi_1(b)\pi_2(y)] + (1 - \delta_1)\delta_2[\pi_1(a) + \pi_1(b)] + \delta_1(1 - \delta_2)\pi_2(y) + \delta_1\delta_2 + \min_{(s_1, s_2) \in [0,1]^2} \left\{ s_1[\delta_1(1 - \delta_2)\pi_2(x)] + s_2[\delta_1(1 - \delta_2)(\pi_2(x) - \delta_2) + s_1s_2\delta_1\delta_2] \right\}.
\]

The partial derivatives of the objective function for the last minimization problem are
\[
\frac{\partial}{\partial s_1} = \delta_1(1 - \delta_2)\pi_2(x) + s_2\delta_1\delta_2 \geq 0,
\]
and
\[
\frac{\partial}{\partial s_2} = -(1 - \delta_1)\delta_2\pi_1(b) - \delta_1\delta_2(1 - s_1) \leq 0,
\]
and thus, $s_1 = 0$ and $s_2 = 1$ form a solution to the minimization problem. Substituting these values into the minimization problem defining $\hat{\alpha}((a, x), (a, y), (b, y))$, yields
\[
\hat{\alpha}((a, x), (a, y), (b, y)) = (1 - \delta_1)(1 - \delta_2)[\pi_1(a) + \pi_1(b)\pi_2(y)] + (1 - \delta_1)\delta_2\pi_1(a) + \delta_1(1 - \delta_2)\pi_2(y),
\]
which implies that
\[
m_{\hat{\alpha}}((a, x), (b, y)) = -m_{\hat{\alpha}}((a, x), (a, y), (b, y)).
\]

Using again the definition of $\hat{\alpha}$, we get that
\[
\hat{\alpha}((a, x), (b, y)) = (1 - \delta_1)(1 - \delta_2)\pi_1(a)\pi_2(x) + (1 - \delta_1)(1 - \delta_2)\pi_1(b)\pi_2(y) + \min_{(s_1, s_2) \in [0,1]^2} \left\{ (1 - \delta_1)\delta_2[\pi_1(a)s_2 + \pi_1(b)(1 - s_2)] + \delta_1(1 - \delta_2)[s_1\pi_2(x) + (1 - s_1)\pi_2(y)] + \delta_1\delta_2[s_1s_2 + (1 - s_1)(1 - s_2)] \right\},
\]
which implies that $m_{\hat{\alpha}}((a, x), (b, y))$ is equal to the minimum value corresponding to the solution of the above minimization problem. Even though the solution to this problem depends on the particular values of the $\delta$’s and $\pi$’s, it is easy to see that the minimum must be strictly positive as long as one of $\pi_1(a)$ or $\pi_2(x)$ lies in the interval $(0, 1)$, in which case $m_{\hat{\alpha}}((a, x), (a, y), (b, y)) < 0$, and thus $\hat{\alpha}$ is not a belief function.

Claim 2. $\hat{\alpha}$ is not a convex capacity.
Proof. \( \hat{\alpha} \) is convex if for all \( A, B \subseteq S_1 \times S_2 \),

\[
\hat{\alpha}(A \cup B) + \hat{\alpha}(A \cap B) \geq \hat{\alpha}(A) + \hat{\alpha}(B).
\]  

(12)

Letting \( A = \{(a, x), (a, y)\} \) and \( B = \{(a, x), (b, y)\} \), the characterization of \( m_{\hat{\alpha}} \) from the proof of Claim 1 implies that equation (12) is equivalent to

\[
m_{\hat{\alpha}}(\{(a, x), (b, y)\}) \geq m_{\hat{\alpha}}(\{(a, x), (b, y)\}).
\]  

(13)

Furthermore, \( m_{\hat{\alpha}}(\{(a, y), (b, y)\}) = \delta_1 (1 - \delta_2) \pi_2(y) \), and if we assume that \( 0 < \pi_1(a) < \pi_1(b) \) and \( \pi_2(x) > \pi_2(y) > 0 \), then

\[
m_{\hat{\alpha}}(\{(a, x), (b, y)\}) = (1 - \delta_1) \delta_2 \pi_1(a) + \delta_1 (1 - \delta_2) \pi_2(y),
\]

which implies that equation (13) can never hold. \( \square \)

Appendix B  Preferences for randomization—an example

This appendix illustrates the results discussed in Section 4.1 using a classic example from Raiffa (1961), which was also analyzed by Eichberger and Kelsey (1996) and Klibanoff (2001), and shows how the SPI can be used to reconcile the various conclusions. The example considers an Ellsberg-style urn containing 100 balls, which can be either red (\( R \)) or green (\( G \)), but where the exact number of red vs. green balls is unknown. A decision-maker faces the state space \( \{R, G\} \) defined by the draw of a ball from this urn, and must choose between two (pure) actions/acts of betting on \( R \) (action \( A \)) or betting on \( G \) (action \( B \))—the decision-maker then wins $100 if the drawn ball is of the correct color, and $0 otherwise. In addition, the decision-maker can commit to an objective randomization over the two bets, where an (independent) fair coin is tossed, and if the outcome is heads (\( H \)), the bet \( A \) is chosen, and if the outcome is tails (\( T \)), the bet \( B \) is chosen. The key to the choice of such a randomization is that the decision-maker can make such a commitment, so that no matter whether the true state is \( R \) or \( G \), the randomization will be implemented.\(^{23}\)

Assuming an Anscombe-Aumann framework where outcomes of acts are allowed to be defined as lotteries over consequences, the random choice can be viewed as being equivalent to the act where in both states \( R \) and \( G \), the resulting outcomes are given by the lottery \( \frac{1}{2}$100 \oplus \frac{1}{2}$0 \). Denoting the corresponding act by \( C \), the choices available to the decision-maker can be summarized as in Figure 3. Assume furthermore that the decision-maker’s preferences have a CEU representation, with beliefs over the state space \( \{R, G\} \) given by a simple capacity \( \alpha \) that is defined by the Möbius transform

\[
m_{\hat{\alpha}}(\{R\}) = m_{\hat{\alpha}}(\{G\}) = \frac{1 - \delta}{2}, \quad m_{\hat{\alpha}}(\{R, G\}) = \delta,
\]

\(^{23}\)As discussed in Raiffa (1961), this can also be interpreted as a situation where a ball is drawn first, and without looking at the color of the ball, a fair coin is tossed and a bet is made based on the outcome of the toss.
with δ ∈ (0, 1). If the decision-maker’s preferences over objective lotteries are represented by the utility function \( u(\$100) = 1, u(\$0) = 0 \), it follows that
\[
u\left(\frac{\$100 + \$0}{2}\right) = \frac{1}{2}.
\]

As discussed in Eichberger and Kelsey (1996) and Klibanoff (2001), this implies a strict preference for the randomization \( C \), since, letting \( U \) denote the CEU of an act with respect to the capacity \( \alpha \), we get
\[
U(C) = \frac{1}{2} > U(A) = U(B) = \frac{1-\delta}{2}u(\$100) + \frac{1-\delta}{2}u(\$0) + \delta \min\{u(\$100), u(\$0)\}.
\]

Note that the Anscombe-Aumann framework used here implies that the values of \( U \) are computed using an iterated integral, where the inside expectation is computed over \( u \) with respect to an objective lottery, whereas the outside expectation is a Choquet integral with respect to \( \alpha \).

Eichberger and Kelsey (1996) argue that it is preferable and more intuitive to model the randomization jointly with all other uncertainty faced by the decision maker, by using a Savage framework that includes the outcomes of the randomization device into the state space. The resulting state space is then given by the Cartesian product \( S := \{R, G\} \times \{H, T\} \), and if we denote the acts of betting on \( R \), betting on \( G \), or randomising by betting on \( R \) if \( H \) and \( G \) if \( T \), by \( D, E \) and \( F \), respectively, we can summarize the resulting outcomes as in Figure 4. As pointed out by Klibanoff (2001), an objective randomization in this setting should require that the probabilities attached to the randomization device are unambiguous and stochastically independent from the rest of the state space. If the decision-maker has CEU preferences, we could then require beliefs to be represented by an independent product of the unambiguous probabilities \( p = \left(\frac{1}{2}, \frac{1}{2}\right) \) over \( \{H, T\} \), and the simple capacity \( \alpha \) over \( \{R, G\} \) introduced above. As shown in Ghirardato (1997, Theorems 2 and 3), there can exist only one such product capacity that is also convex, and this capacity must equal the Möbius product of \( p \).
and \( \alpha \). Denote this Möbius product by \( \bar{\alpha} \), and its Möbius transform by \( m_{\bar{\alpha}} \). Then, since the Möbius transform of \( p \) assigns zero weight to the set \( \{H, T\} \), \( m_{\bar{\alpha}} \) must satisfy

\[
m_{\bar{\alpha}}(\text{any singleton}) = \frac{1 - \delta}{4}, \quad m_{\bar{\alpha}}(\{RH, GH\}) = m_{\bar{\alpha}}(\{RT, GT\}) = \frac{\delta}{2},
\]

and must be equal to zero on all other subsets of \( S = \{R, G\} \times \{H, T\} \). Hence, using the utility function \( u(\$100) = 1, u(\$0) = 0 \), and denoting the CEU of the choices \( D, E \) and \( F \) with respect to \( \bar{\alpha} \) by \( \bar{U} \), we get

\[
\bar{U}(D) = \bar{U}(E) = \bar{U}(F) = \frac{1 - \delta}{2},
\]

so the decision-maker is indifferent between the initial two bets and the randomization. In fact, Eichberger and Kelsey (1996) show that such an indifference must always hold as long as the product capacity over the joint state space is convex, and the outcomes of the actions over which the randomization takes place do not depend on the state space of the unambiguous randomization device.

One way to interpret the CEU corresponding to \( F \) is to note that the corresponding Choquet integral can be computed as

\[
\bar{U}(F) = \frac{1 - \delta}{2} + m_{\bar{\alpha}}(\{RH, GH\}) \min\{u_F(RH), u_F(GH)\} + m_{\bar{\alpha}}(\{RT, GT\}) \min\{u_F(RT), u_F(GT)\}
\]

\[
= \frac{1 - \delta}{2} + \frac{\delta}{2} \min\{1, 0\} + \frac{\delta}{2} \min\{0, 1\} = \frac{1 - \delta}{2},
\]

where \( u_F(s) \) is the utility of the outcome corresponding to state \( s \in \{R, G\} \times \{H, T\} \) when act \( F \) is chosen. Hence, if the decision-maker chooses the randomization, the last two components of the integral implicitly allow for a possible correlation between the randomization device and the color of the ball. This is analogous to the potential correlations between two opponents in a game as discussed in Section 3.1, and was also pointed out by Klibanoff (2001, p. 616), who notes that such CEU preferences “force a range of possible correlations (which are then viewed pessimistically since they are another source of uncertainty) between the device and the rest of the state space.” Klibanoff (2001) then proposes a preference-based condition for stochastic independence of a randomization device, in the presence of which CEU preferences are shown to collapse to expected utility, and concludes that as a consequence of this result, CEU preferences are unable to model a stochastically independent randomization device.

As the SPI was constructed to eliminate the type of correlation pointed out by Klibanoff (2001), we would expect that preferences represented by the SPI should satisfy Klibanoff’s condition for stochastic independence. That this must indeed be the case follows from our Proposition 1, together with Theorem 1 in Klibanoff (2001). By Theorem 1 in Klibanoff (2001), for preferences that can be represented by MEU, the stochastic independence condition is equivalent to the set of beliefs corresponding to the MEU representation being given by the independent products of a unique probability distribution over the randomization space, and a set of distributions over the space of ambiguous
uncertainty. The SPI is equivalent to such a representation, as shown by Proposition 1. In the example above, the core of the unambiguous distribution \( p \) is \( C(p) = \{p\} \), and the SPI is equivalent to the MEU over the set of beliefs \( \{p\} \otimes C(\alpha) \), which is exactly of the form required by Klibanoff’s theorem. Hence, not only must preferences represented by the SPI satisfy Klibanoff’s stochastic independence condition, but they should also allow for strict preferences for randomization. This can be verified by computing the SPI for the acts \( D, E \) and \( F \) described in Figure 4. Denoting the corresponding utility functions by \( u_D, u_E \) and \( u_F \), we get

\[
\int_S u_D(p \otimes \alpha) = \min_{\sigma \in [0,1]} \left\{ (1 - \delta) \sum_{s \in S} \frac{u_D(s)}{4} + \delta \left[ \sigma(u_D(RH) + u_D(RT)) + (1 - \sigma)(u_D(GH) + u_D(GT)) \right] \right\}
\]

\[
= \frac{1 - \delta}{2} + \frac{\delta}{2} \min_{\sigma \in [0,1]} \{2\sigma\} = \frac{1 - \delta}{2},
\]

and analogously,

\[
\int_S u_E(p \otimes \alpha) = \min_{\sigma \in [0,1]} \left\{ (1 - \delta) \sum_{s \in S} \frac{u_E(s)}{4} + \delta \left[ \sigma(u_E(RH) + u_E(RT)) + (1 - \sigma)(u_E(GH) + u_E(GT)) \right] \right\}
\]

\[
= \frac{1 - \delta}{2} + \frac{\delta}{2} \min_{\sigma \in [0,1]} \{2(1 - \sigma)\} = \frac{1 - \delta}{2}.
\]

Furthermore, for the act \( F \) corresponding to the randomization, the SPI is

\[
\int_S u_F(p \otimes \alpha) = \min_{\sigma \in [0,1]} \left\{ (1 - \delta) \sum_{s \in S} \frac{u_F(s)}{4} + \delta \left[ \sigma(u_F(RH) + u_F(RT)) + (1 - \sigma)(u_F(GH) + u_F(GT)) \right] \right\}
\]

\[
= \frac{1 - \delta}{2} + \frac{\delta}{2} \min_{\sigma \in [0,1]} \{\sigma + (1 - \sigma)\} = \frac{1}{2}.
\]

Hence, the strict preference for randomization reappears if the SPI is used to compute the expected utility over the product space, even when the acts are defined in a Savage framework.

**Appendix C  Four examples illustrating properties of EBF**

**Example 1.** Consider a version of the matching pennies game as described in Figure 5, which is also analyzed in Eichberger and Kelsey (2000). This game has a unique NE in mixed strategies, where each player assigns probability \( \frac{1}{2} \) to each one of his actions.
Similarly, player 2’s expected utility is given by

\[ U_2(p, q) = \min_{\sigma \in [0, 1]} \left\{ (1 - \epsilon_2)[p(\epsilon_2) + (1 - p)(1 - \epsilon_2)] + \epsilon_1[p\epsilon_1 + (1 - p)(1 - \epsilon_1)] \right\} \]

\[ = \begin{cases} 
(1 - \epsilon_2)(1 - q) + \epsilon_1 + p[2q(1 - \epsilon_2) - 1], & \text{if } p \leq \frac{1}{2}, \\
(1 - \epsilon_2)(1 - q) + \epsilon_1 + p[2q(1 - \epsilon_2) - 1], & \text{if } p > \frac{1}{2}.
\end{cases} \]

Similarly, player 2’s expected utility is given by

\[ U_2(q, p) = \min_{\gamma \in [0, 1]} \left\{ (1 - \epsilon_2)[p(\epsilon_2) + (1 - p)(1 - \epsilon_2)] + \epsilon_1[p\epsilon_1 + (1 - p)(1 - \epsilon_1)] \right\} \]

\[ = \begin{cases} 
(1 - \epsilon_2)(1 - q) + \epsilon_1 + q[2p(1 - \epsilon_2) - 1], & \text{if } q \leq \frac{1}{2}, \\
(1 - \epsilon_2)(1 - q) + \epsilon_1 + q[2p(1 - \epsilon_2) - 1], & \text{if } q > \frac{1}{2}.
\end{cases} \]

The resulting best response correspondences for the two players are

\[ BR_1(q) = \begin{cases} 
p = 0, & \text{if } q < \frac{1 - 2\epsilon_2}{2(1 - \epsilon_2)}, \\
p \in [0, \frac{1}{2}], & \text{if } q = \frac{1 - 2\epsilon_2}{2(1 - \epsilon_2)}, \\
p = \frac{1}{2}, & \text{if } q \in \left( \frac{1 - 2\epsilon_2}{2(1 - \epsilon_2)}, \frac{1}{2(1 - \epsilon_2)} \right), \\
p \in \left[ \frac{1}{2}, 1 \right], & \text{if } q = \frac{1}{2}, \\
p = 1, & \text{if } q > \frac{1}{2(1 - \epsilon_2)}.
\end{cases} \]

\[ BR_2(p) = \begin{cases} 
q = 1, & \text{if } p < \frac{1 - 2\epsilon_1}{2(1 - \epsilon_1)}, \\
q \in \left[ \frac{1}{2}, 1 \right], & \text{if } p = \frac{1 - 2\epsilon_1}{2(1 - \epsilon_1)}, \\
q = \frac{1}{2}, & \text{if } p \in \left( \frac{1 - 2\epsilon_1}{2(1 - \epsilon_1)}, \frac{1}{2(1 - \epsilon_1)} \right), \\
q \in [0, \frac{1}{2}], & \text{if } p = \frac{1}{2}, \\
q = 0, & \text{if } p > \frac{1}{2(1 - \epsilon_1)}.
\end{cases} \]
An analysis of the best responses then implies that the game has a unique EBF in mixed strategies, defined by $p = q = \frac{1}{2}$. While this result seems to be equivalent to the NE of the game where no ambiguity is present, the crucial difference between the EBF and the NE is that even though the predicted strategies are the same, as long as $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, in the EBF the equilibrium strategies are strictly optimal for both players. Thus, even with very small degrees of ambiguity, each player has a unique strictly optimal strategy in equilibrium, is in contrast to NE, where any completely mixed equilibrium strategy can never be strictly optimal.

It is also interesting to compare the predictions of the unique EBF with those of the EuA of Eichberger and Kelsey (2000). While the resulting equilibria are essentially equivalent, since EuA are given by any capacities that assign equal weights to the two actions $H$ and $T$ (which, in the EBF, correspond to the belief functions induced by various values of $\varepsilon_1$ and $\varepsilon_2$, together with the equilibrium strategies $p = q = \frac{1}{2}$), the associated interpretations are different. For EuA, the prediction is that both players’ equilibrium beliefs assign equal weights to both actions of their opponents, and hence, any one of their own pure actions yields the same payoff, which implies that either one of the pure actions may be played in equilibrium (as mixed strategies are not feasible objects of choice). In contrast, EBF predicts that in the unique equilibrium each player must play the unique, strictly optimal, mixed strategy that provides an optimal hedge against the perceived uncertainty regarding the opponent’s strategy, by assigns equal weight of $\frac{1}{2}$ to each available action.

**Example 2.** To illustrate the flexibility of EBF in characterizing equilibrium behavior and beliefs, we expand the matching pennies game from Example 1 by adding a strictly dominated action $E$ for player 2, as illustrated in Figure 6. In an EBF, the beliefs of player 1 regarding the possible actions of player 2 are described by a belief function that can be characterized by a distribution $p_2$ over strategic types, and a mixed strategy $\pi_2$ for the rational type of player 2. Since player 2 now has three (pure) actions, $p_2$ is defined over a five-element set, as the action set $\{H, T, E\}$ has four non-empty, non-singleton subsets. The action set of player 1 is unchanged, so any belief function over $\{H, T\}$ must be a simple capacity, as in Example 1.

Consider first a theory of play that specifies that no player will choose a strictly dominated action, not even as the result of a mistake or tremble. If this assumption is common knowledge, any corresponding $p_2$ that is part of an EBF must assign probability zero to every subset of 2’s action
set \( \{H, T, E\} \) that contains the strictly dominated action \( E \), so only the non-rational type defined by \( \{H, T\} \) is assigned strictly positive probability. Similarly, any equilibrium mixed strategy \( \pi_2 \) must assign probability zero to \( E \). The analysis from Example 1 then shows that every EBF, irrespective of the associated distributions over strategic types, requires the equilibrium mixed strategies \( \pi_1 \) and \( \pi_2 \) to randomize with equal probabilities over \( H \) and \( T \), so \( \pi_1 = \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( \pi_2 = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \).

An alternative theory could be based on the assumption that mistakes are just involuntary trembles that are unrelated to payoffs, so that any such mistake could lead to some arbitrary distribution over the action set \( \{H, T, E\} \). Such a situation could be captured by an EBF where \( p_2(\{H, T, E\}) = \epsilon_2 \), and the probability of a rational type of player 2 is just \( 1 - \epsilon_2 \), which implies that the actions of 2 are described by a simple capacity over \( \{H, T, E\} \). The resulting set of EBF can be derived analogous to Example 1, noting that the mixed strategy of player 2 that achieves the worst-case payoff for player 1 assigns probability one to \( E \). It can then be shown that as long as \( \epsilon_2 > \frac{1}{100} \), the mixed strategies \( \pi_1 \) and \( \pi_2 \) in a resulting EBF must involve player 1 choosing the pure action \( H \) and player 2 choosing the pure action \( T \).

Clearly, the two cases discussed above are just possible examples that illustrate the different types of belief functions that can define an EBF. The point of this analysis is not that the resulting EBF are of particular interest, but that different assumptions regarding strategic uncertainty can easily be described by various belief functions, and can lead to distinct equilibrium predictions.

\textbf{Example 3.} Consider next the three-player game described in Figure 7. This game is a slightly modified version of an example analyzed in Eichberger and Kelsey (2000). Their objective in considering the game is to present an EuA where two players may have inconsistent beliefs regarding the actions of a third player. Since EBF by definition requires consistent beliefs about the strategies of such a third player, clearly, no EBF will include such inconsistent beliefs, and hence, any such equilibrium cannot define an EBF. Our main goal in presenting the example is to show how limit EBF can be used as a refinement of NE.

Since, as in Example 1, each player has only two available actions, any belief function defined on the action space of a player \( i \) can equivalently be expressed as a simple capacity induced by a probability distribution \( p_i = (1 - \epsilon_i, \epsilon_i) \) over the player’s types, together with a mixed strategy \( \pi_i \). We will assume that the probabilities \( \epsilon_i \) assigned to the irrational types are equal across players and given by some \( \epsilon \in (0, 1) \), and denote the mixed strategies of the three players by \( \pi_1 \equiv (p, 1 - p) \), \( \pi_2 \equiv (q, 1 - q) \) and \( \pi_3 \equiv (r, 1 - r) \), where \( p, q \) and \( r \) are the probabilities assigned to the actions \( T \), \( L \) and \( W \), respectively.

If there is no ambiguity, the set of NE of the game consists of the following mixed strategy combinations:
Figure 7: The three-player game from Example 3.

- $p = 0$, $q = 0$ and $r \in [0, 1]$;
- $p \in (0, 1]$, $q = 0$ and $r = 1$;
- $p = 0$, $q \in (0, 1]$ and $r = 0$; or
- $p = q \in (0, 1]$ and $r = \frac{1}{2}$.

When $\varepsilon > 0$, so the players’ beliefs contain some degree of ambiguity, we will see that some (but not all) of these NE also define EBF, but that there exist mixed strategy profiles that yield an EBF without being NE. As the degree of ambiguity $\varepsilon$ converges to zero, the set of EBF converges to a strict subset of the set of NE, as shown in Proposition 7.

With a positive degree of ambiguity $\varepsilon$, the expected utility of player 1 is given by

$$U_1(p, q, r) = \min_{\sigma \in [0, 1], \mu \in [0, 1]} \{(1 - \varepsilon)^2 [pqr + (1 - p)q(1 - r)] + (1 - \varepsilon)\varepsilon[pq\mu + (1 - p)q(1 - \mu)]$$
$$+ \varepsilon(1 - \varepsilon)[p\sigma r + (1 - p)\sigma(1 - r)] + \varepsilon^2[p\sigma\mu + (1 - p)\sigma(1 - \mu)]\}.$$

Since the expression that is being minimized over $(\sigma, \mu) \in [0, 1]^2$ is linear in $\sigma$ for given $\mu$, and linear in $\mu$ for given $\sigma$, the minimum must be attained at the vertices of $[0, 1]^2$, which implies that

$$U_1(p, q, r) = \begin{cases} (1 - \varepsilon)^2 q(1 - r) + p[(1 - \varepsilon)^2 q(2r - 1) + (1 - \varepsilon)\varepsilon q], & \text{if } p \leq \frac{1}{2}, \\ (1 - \varepsilon)^2 q(1 - r) + (1 - \varepsilon)\varepsilon q + p[(1 - \varepsilon)^2 q(2r - 1) - (1 - \varepsilon)\varepsilon q], & \text{if } p > \frac{1}{2}. \end{cases}$$

The best response of player 1 is then given by

$$BR_1(q, r) = \begin{cases} p \in [0, 1], & \text{if } q = 0, \\ p = 0, & \text{if } q \neq 0 \text{ and } r < \frac{1 - 2\varepsilon}{2(1 - \varepsilon)}, \\ p \in [0, \frac{1}{2}], & \text{if } q \neq 0 \text{ and } r = \frac{1 - 2\varepsilon}{2(1 - \varepsilon)}, \\ p = \frac{1}{2}, & \text{if } q \neq 0 \text{ and } r \in \left(\frac{1 - 2\varepsilon}{2(1 - \varepsilon)}, \frac{1}{2(1 - \varepsilon)}\right), \\ p \in \left[\frac{1}{2}, 1\right], & \text{if } q \neq 0 \text{ and } r = \frac{1}{2(1 - \varepsilon)}, \\ p = 1, & \text{if } q \neq 0 \text{ and } r > \frac{1}{2(1 - \varepsilon)}. \end{cases}$$
Analogously, the expected utility of player 2 is

\[
U_2(q, p, r) = \min_{\gamma \in [0,1], \mu \in [0,1]} \{ (1 - \varepsilon)^2(p(1-q)r + pq(1-r)) + (1 - \varepsilon)\varepsilon[p(1-q)\mu + pq(1-\mu)] \\
+ \varepsilon (1 - \varepsilon) [\gamma(1-q)r + \gamma q(1-r)] + \varepsilon^2[\gamma(1-q)\mu + \gamma q(1-\mu)] \},
\]

which, after solving the associated minimization problem, yields

\[
U_2(q, p, r) = \begin{cases} 
(1 - \varepsilon)^2 pr + q[(1 - \varepsilon)^2 p(1-2r) + (1 - \varepsilon)\varepsilon p], & \text{if } q \leq \frac{1}{2}, \\
(1 - \varepsilon)^2 pr + (1 - \varepsilon)\varepsilon p + q[(1 - \varepsilon)^2 p(1-2r) - (1 - \varepsilon)\varepsilon p], & \text{if } q > \frac{1}{2}.
\end{cases}
\]

The resulting best response for player 2 is then

\[
BR_2(p, r) = \begin{cases} 
q \in [0,1], & \text{if } p = 0, \\
q = 1, & \text{if } p \neq 0 \text{ and } r < \frac{1-2\varepsilon}{2(1-\varepsilon)}, \\
q \in \left[\frac{1}{2},1\right], & \text{if } p \neq 0 \text{ and } r = \frac{1-2\varepsilon}{2(1-\varepsilon)}, \\
q = \frac{1}{2}, & \text{if } p \neq 0 \text{ and } r \in \left(\frac{1-2\varepsilon}{2(1-\varepsilon)}, \frac{1}{2(1-\varepsilon)}\right), \\
q \in [0,\frac{1}{2}], & \text{if } p \neq 0 \text{ and } r = \frac{1}{2(1-\varepsilon)}, \\
q = 0, & \text{if } p \neq 0 \text{ and } r > \frac{1}{2(1-\varepsilon)}.
\end{cases}
\]

Finally, player 3’s expected utility is

\[
U_3(r, p, q) = \min_{\gamma \in [0,1], \sigma \in [0,1]} \{ (1 - \varepsilon)^2[p(1-q)r + (1-p)q(1-r)] + (1 - \varepsilon)\varepsilon[p(1-\sigma)r + (1-p)\sigma(1-r)] \\
+ \varepsilon(1 - \varepsilon) [\gamma(1-q)r + (1-\gamma)q(1-r)] + \varepsilon^2[\gamma(1-\sigma)r + (1-\gamma)\sigma(1-r)] \},
\]

and solving the minimization problem yields

\[
U_3(r, p, q) = \begin{cases} 
(1 - \varepsilon)^2(1-p)q + r[(1 - \varepsilon)(p-q) + \varepsilon], & \text{if } q \leq 1-p \text{ and } r \leq (1-\varepsilon)q, \\
(1 - \varepsilon)^2(1-p)q + (1 - \varepsilon)\varepsilon q \\
+ r[(1 - \varepsilon)(p-q)], & \text{if } q \leq 1-p \text{ and } r \in ((1-\varepsilon)q, (1-\varepsilon)(1-p) + \varepsilon], \\
(1 - \varepsilon)^2(1-p)q + (1 - \varepsilon)\varepsilon(1-p + q) \\
+ \varepsilon^2 + r[(1 - \varepsilon)(p-q) - \varepsilon], & \text{if } q \leq 1-p \text{ and } r > (1-\varepsilon)(1-p) + \varepsilon, \\
(1 - \varepsilon)^2(1-p)q + r[(1 - \varepsilon)(p-q) + \varepsilon], & \text{if } q > 1-p \text{ and } r \leq (1-\varepsilon)(1-p), \\
(1 - \varepsilon)^2(1-p)q + (1 - \varepsilon)\varepsilon(1-p) \\
+ r[(1 - \varepsilon)(p-q)], & \text{if } q > 1-p \text{ and } r \in ((1-\varepsilon)(1-p), (1-\varepsilon)q + \varepsilon], \\
(1 - \varepsilon)^2(1-p)q + (1 - \varepsilon)\varepsilon(1-p + q) \\
+ \varepsilon^2 + r[(1 - \varepsilon)(p-q) - \varepsilon], & \text{if } q > 1-p \text{ and } r > (1-\varepsilon)q + \varepsilon.
\end{cases}
\]
Player 3’s best response is then given by

\[
BR_3(p, q) = \begin{cases} 
  r = 1, & \text{if } q < p - \frac{\epsilon}{1 - \epsilon}, \\
  r \in [(1 - \epsilon)(1 - p) + \epsilon, 1], & \text{if } q = p - \frac{\epsilon}{1 - \epsilon} \text{ and } q \leq 1 - p, \\
  r \in [(1 - \epsilon)q + \epsilon, 1], & \text{if } q = p - \frac{\epsilon}{1 - \epsilon} \text{ and } q > 1 - p, \\
  r = (1 - \epsilon)(1 - p) + \epsilon, & \text{if } q \in (p - \frac{\epsilon}{1 - \epsilon}, p) \text{ and } q \leq 1 - p, \\
  r = (1 - \epsilon)q + \epsilon, & \text{if } q \in (p - \frac{\epsilon}{1 - \epsilon}, p) \text{ and } q > 1 - p, \\
  r \in [(1 - \epsilon)q, (1 - \epsilon)(1 - p) + \epsilon], & \text{if } q = p \text{ and } q \leq 1 - p, \\
  r \in [(1 - \epsilon)(1 - p), (1 - \epsilon)q + \epsilon], & \text{if } q = p \text{ and } q > 1 - p, \\
  r = (1 - \epsilon)q, & \text{if } q \in (p, p + \frac{\epsilon}{1 - \epsilon}) \text{ and } q \leq 1 - p, \\
  r = (1 - \epsilon)(1 - p), & \text{if } q \in (p, p + \frac{\epsilon}{1 - \epsilon}) \text{ and } q > 1 - p, \\
  r \in [0, (1 - \epsilon)q], & \text{if } q = p + \frac{\epsilon}{1 - \epsilon} \text{ and } q \leq 1 - p, \\
  r \in [0, (1 - \epsilon)(1 - p)], & \text{if } q = p + \frac{\epsilon}{1 - \epsilon} \text{ and } q > 1 - p, \\
  r = 0, & \text{if } q > p + \frac{\epsilon}{1 - \epsilon}. 
\end{cases}
\]

While this best response looks rather complex, it can easily be visualized using a two-dimensional diagram with \( p \) on the horizontal axis and \( q \) on the vertical axis, as the main regions on this diagram that describe the best responses are delineated by the points where \( q = 1 - p, q = p - \frac{\epsilon}{1 - \epsilon}, q = p \) and \( q = p + \frac{\epsilon}{1 - \epsilon} \).

Analyzing the best responses of all three players then implies that the set of EBF consists of the following mixed strategy profiles (as long as \( \epsilon \) is sufficiently small, i.e., for \( \epsilon \) approximately less than 0.3):

- \( p = 0, q = 0 \) and \( r \in [0, 1] \);
- \( p \in \left( \frac{\epsilon}{1 - \epsilon}, 1 \right], q = 0 \) and \( r = 1 \);
- \( p = 0, q \in \left( \frac{\epsilon}{1 - \epsilon}, 1 \right] \) and \( r = 0 \);
- \( p = \frac{\epsilon}{1 - \epsilon}, q = 0 \) and \( r \in [1 - \epsilon, 1] \);
- \( p \in \left( 0, \frac{\epsilon}{1 - \epsilon} \right), q = 0 \) and \( r = (1 - \epsilon)(1 - p) + \epsilon \);
- \( p = 0, q = \frac{\epsilon}{1 - \epsilon} \) and \( r \in [0, \epsilon] \);
- \( p = 0, q \in \left( 0, \frac{\epsilon}{1 - \epsilon} \right) \) and \( r = (1 - \epsilon)q \);
- \( p = q = \frac{1}{2} \) and \( r \in \left[ \frac{1 - \epsilon}{2}, \frac{1 + \epsilon}{2} \right] \);
These EBF strategy profiles are also best visualized using a two-dimensional $p$-$q$ coordinate system. As previously indicated, for strictly positive values of $\epsilon$, there exist EBF strategy profiles that are not also NE. Furthermore, the set of limit EBF as $\epsilon \rightarrow 0$ is a strict subset of the set of NE: The EBF described in the first three bullet points converge to the NE from the first three bullet points describing the set of NE; the EBF from the fourth and fifth bullet points converge to $p = q = 0$ and $r = 1$, whereas those from the sixth and seventh bullet points converge to $p = q = r = 0$; finally, the EBF from the last three bullet points all converge to the mixed strategy profile $p = q = r = \frac{1}{2}$, which is one of the NE in mixed strategies. Note, however, that none of the NE with $r = \frac{1}{2}$ and $p = q \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ can be approximated by EBF with strictly positive $\epsilon$, and hence, such NE would not be considered robust to the introduction of ambiguity.

Example 4. This example incorporates strategic uncertainty into a simple costly voting game inspired by Osborne (2004, Exercise 34.2). Consider a game where two candidates, $A$ and $B$, compete in an election for the votes of three citizens, so that the players in the game are the three voters. Voting is not compulsory, and carries a cost $c \in (0, 1)$ that is incurred by any citizen who chooses to cast a vote. It is common knowledge that voters 1 and 2 prefer candidate $A$ to win, and that voter 3 prefers candidate $B$. Furthermore, if a citizen decides to vote, he votes for his most preferred candidate. A citizen who abstains receives a payoff of 2 if his preferred candidate receives a majority of the submitted votes, a payoff of 1 if there is a tie, and a payoff of 0 if his preferred candidate loses the vote. A citizen who votes receives the same payoffs minus the voting cost $c$. Denoting by $V$ and $N$ the actions of voting or abstaining, the resulting game can be described as in Figure 8, where voter 1 chooses a row, voter 2 a column, and voter 3 a table.

As in the previous example, there are only two actions for each player, so we can describe a belief function over the action space by letting $\epsilon \in (0, 1)$ denote the probability of an irrational type who plays an arbitrary strategy in $\Delta(V, N)$, where $\epsilon$ is assumed to be equal across the three players, and denoting the individual mixed strategies of the players by $\pi_1 = (p, 1 - p)$, $\pi_2 = (q, 1 - q)$ and
\( \pi_3 = (r, 1-r) \), where \( p, q \) and \( r \) are the probabilities that the respective players choose to vote.

With no ambiguity, when \( \varepsilon = 0 \), the resulting game does not have any NE in pure strategies, but possesses the following three NE in mixed strategies:

- \( p = 1, q = 1-c, r = c \);
- \( p = 1-c, q = 1, r = c \); and
- \( p = \sqrt{1-c}, q = \sqrt{1-c}, r = 1 - \sqrt{1-c} \).

In the case of a positive degree of ambiguity \( \varepsilon \in (0, 1) \), the approach used in Example 3 can be applied to derive the following best responses for the three players:

\[
BR_1(q,r) = \begin{cases} 
  p = 1, & \text{if } (1-\varepsilon)^2q(1-r) < 1-c, \\
  p \in [0,1], & \text{if } (1-\varepsilon)^2q(1-r) = 1-c, \\
  p = 0, & \text{if } (1-\varepsilon)^2q(1-r) > 1-c;
\end{cases}
\]

\[
BR_2(p,r) = \begin{cases} 
  q = 1, & \text{if } (1-\varepsilon)^2p(1-r) < 1-c, \\
  q \in [0,1], & \text{if } (1-\varepsilon)^2p(1-r) = 1-c, \\
  q = 0, & \text{if } (1-\varepsilon)^2p(1-r) > 1-c;
\end{cases}
\]

\[
BR_3(p,q) = \begin{cases} 
  r = 1, & \text{if } (1-\varepsilon)^2pq + \varepsilon(1-\varepsilon)(p+q) + \varepsilon^2 < 1-c, \\
  r \in [0,1], & \text{if } (1-\varepsilon)^2pq + \varepsilon(1-\varepsilon)(p+q) + \varepsilon^2 = 1-c, \\
  r = 0, & \text{if } (1-\varepsilon)^2pq + \varepsilon(1-\varepsilon)(p+q) + \varepsilon^2 > 1-c.
\end{cases}
\]

In contrast to the case with no ambiguity, when \( \varepsilon > 0 \), the game can also possess EBF in pure strategies. With extreme ambiguity, when \( \varepsilon = 1 \), the only EBF involves the action profile \((V, V, N)\). Furthermore, even when \( \varepsilon \in (0, 1) \), there exist values for the parameters \( \varepsilon \) and \( c \) under which pure strategy EBF exist. In particular, as long as the inequalities \( c > 1 - \varepsilon \) and \( c > 2\varepsilon - \varepsilon^2 \) both hold (which is a non-empty set), the action profiles \((V, N, N)\) and \((N, V, N)\) constitute EBF. Given these equilibrium action profiles, the candidate who has an ex-ante majority given the voters’ preferences (candidate \( A \)) wins with probability one, and the voting costs are minimized, as only one of the citizens casts his vote (which also yields lower expected total voting costs than any of the mixed strategy NE without ambiguity). Hence, the presence of strategic uncertainty is socially beneficial, as it results in the most efficient voting outcome at the lowest expected cost.

In addition to the pure strategy EBF discussed above, when \( \varepsilon \) is sufficiently small relative to \( c \) (specifically, two conditions that must be satisfied are \( c < 1 - \varepsilon \) and \( c > 2\varepsilon - \varepsilon^2 \)), the game also possesses the following EBF in mixed strategies:
\[ p = 1, q = \frac{1-c-\varepsilon}{1-c}, r = 1 - \frac{1-c}{(1-\varepsilon)^2}; \]

\[ p = \frac{1-c-\varepsilon}{1-\varepsilon}, q = 1, r = 1 - \frac{1-c}{(1-\varepsilon)^2}; \text{ and} \]

\[ p = \frac{\sqrt{1-c-\varepsilon}}{1-\varepsilon}, q = \frac{\sqrt{1-c-\varepsilon}}{1-\varepsilon}, r = 1 - \frac{1-c}{(1-\varepsilon)(\sqrt{1-c-\varepsilon})}. \]

These mixed strategy profiles can easily be seen to converge to the mixed strategy NE of the game without ambiguity when \( \varepsilon \rightarrow 0 \). Hence, all the NE of the game without ambiguity are robust to the introduction of a small amount of ambiguity. Moreover, the voting probabilities in these mixed strategy EBF are decreasing in \( \varepsilon \) for every player. Hence, an increase in the degree of ambiguity leads to a decrease in the total expected voting cost, and furthermore, the total expected voting cost associated with each of these mixed strategy EBF is always lower than for the corresponding NE that they approximate.

\section*{References}


