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MONEY, CAPITAL AND EXCHANGE RATE FLUCTUATIONS★

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ABSTRACT

We explore how the informational frictions underlying monetary exchange affect international exchange rate dynamics. Using a two-country, two-sector model, we show that information frictions imply a particular restriction on domestic price dynamics and hence on international nominal and real exchange rate determination. Furthermore, if capital is utilized as a factor of production in both production sectors, then there is a further restriction on asset pricing relations (money and capital). As a result, monetary and real outcomes become interdependent in the model. Our perfectly flexible price model is capable of producing endogenously rigid international relative prices in response to technology and monetary shocks. The model is capable of accounting for the empirical regularities that the real and nominal exchange rates are more volatile than U.S. output, and that the two are positively and perfectly correlated. The model is also consistent with other standard real business cycle facts for the U.S.

JEL CODES: E31; E32; E43; E44
KEYWORDS: Search-theoretic Money; Open Economy; Exchange Rate Puzzle

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1. Introduction

It is well known that the real and nominal exchange rates of the world's largest economies are very volatile and persistent. Moreover, these two time series are perfectly and positively correlated. The seminal work of Chari, Kehoe, and McGrattan [2002] explored whether these empirical regularities could be understood in the context of a standard two country real business cycle model with sticky prices. They concluded that such models can account for the volatility of the exchange rates, but not their persistence. Ad-hoc sticky price models are able to generate volatile real and nominal exchange rate processes, because, by assumption prices are made to not adjust too quickly to aggregate shocks. In an open economy, the nominal exchange rate and therefore, the real exchange rate, have to overreact. This is a manifestation of the textbook Dornbusch [1976] exchange rate overshooting hypothesis.

The key ingredient in modern monetary theory, and in our model, is a notion of *anonymity* of traders. Anonymity is a term for: (i) The lack of, or, imperfect record-keeping of individual trader's histories; (ii) Nonexistence of public communication of individual trading histories; and (iii) Lack of enforcement of private contracts. Given this assumption of anonymity and coupled with a random market participating (or meeting) environment (which gives rise to a lack of double coincidence of wants), one can thus rationalize an equilibrium theory for money. Money in this type of environment is thus a medium of exchange and a store of value (i.e. serves a precautionary asset function). In contrast, existing monetary business-cycle models introduce money in more reduced-form ways using either money-in-the-utility (MIU) or cash-in-advance constraints (CIA) [e.g. Chari, Kehoe, and McGrattan, 2002; Schlagenhauf and Wrase, 1995]. These are not innocuous modelling choices. We show that anonymity, and therefore deeper monetary friction, matters for the dynamics of relative prices domestically and internationally.

In this paper, we examine whether a flexible price, two-country, search theoretic model of money is able to account for the empirical regularities observed in U.S. real and nominal exchange rates.[1]

[1] Alessandria [2009] also departs from the standard Walrasian business cycle framework. He develops a model where in each country, there is a "large family" consisting of a continuum of worker-shoppers who engage in noisy search (*i.e.* the number of price quotes each shopper faces is a random variable) *à la* Burdett and Judd [1983]. The shoppers aim to find the "best" price of a single unit of a good offered by domestic or foreign firms. The opportunity cost of search is a function of the worker-shopper's forgone real wage. Because of shoppers' objective to find the best quote and such search is noisy, firms can price discriminate across markets. The equilibrium distributions of prices will be different across countries as a function of international relative real wages. Given relative aggregate country-specific technology and/or taste shocks, which change cross-country relative real wages, the distribution of prices in the home country will shift relative to that in the foreign country. This results in an endogenous deviation from the law of one price, and hence large cross-country relative price fluctuations at the both the aggregate and disaggregated levels.

In contrast to Alessandria [2009], our key friction is a monetary one and arises only in a specific decentralized sector of each country. There is no cross-country search by buyers in our model. Our centralized market (CM), where international trade and asset flows determine the nominal and real exchange rates, is similar to standard Walrasian international business cycle models. This feature facilitates closer comparison with existing international monetary models [e.g. Chari, Kehoe, and McGrattan, 2002; Schlagenhauf and Wrase, 1995]. Moreover, given that we have a monetary model, we can also have something to say about the empirical regularity that the international real and nominal exchange rates for the U.S. are perfectly and positively correlated.

We consider a two-country stochastic version of Lagos and Wright [2005] and Aruoba, Waller, and Wright [2009], where there exist two sectors or sequential submarkets within each period. These sectors comprise a decentralized market (DM) with anonymous (or partially anonymous) trading, and, a centralized Walrasian market (CM). We assume that international trade and asset flows occur in the model's frictionless CM. The CM assumption allows direct comparisons with existing international monetary business cycle models with flexible prices [e.g. Schlagenhauf and Wrase, 1995] and models with sticky prices [e.g. Chari, Kehoe, and McGrattan, 2002], while providing a deeper foundation for money and an alternative equilibrium restriction on pricing processes. Following Aruoba, Waller, and Wright [2009], we allow for installed capital in each centralized market (CM) to be a productive input for sellers in each subsequent decentralized market (DM). This aspect of "capital complementarity" generates an equilibrium linkage between inflation and real economic activity across the DM and CM.

There are two key mechanisms at work in this model which help amplify and propagate international business cycle shocks. The first mechanism is anonymity. This friction induces asset market incompleteness in the sense that individuals are unable to fully insure against their stochastic trading opportunities in the DM. In our benchmark model with logarithmic utility functions and DM price taking, we can contrast our DM *equilibrium pricing condition* with a standard model's cash-in-advance (CIA) *constraint*. In particular, the CIA constraint appears as an *ad-hoc*, reduced-form, special case of our equilibrium condition. Since our DM equilibrium pricing condition relates to buyers' and sellers' primitive preferences and technologies, then, money supply and technology shocks become directly encoded in the DM equilibrium pricing condition. Thus, depending on the DM Walrasian pricing protocol (or sharing rule in a bargaining version), domestic prices need not respond by as much to home technology and money supply growth shocks. This would also be true in the foreign country. Thus in the equilibrium of our calibrated model, we show that relative aggregate prices across countries do not respond as much to country-specific technology or money supply growth shocks. This explains why the model is able to account for the volatility of the exchange rates.

The assumption of anonymity also introduces a liquidity premium for money which arises in equilibrium asset-pricing Euler equations. This additional liquidity premium, reflecting the equilibrium role of money as medium of exchange and store of value, depends on agents' risk aversion and production technology. In this paper (section 4.1), we demonstrate how this relates to the persistence of relative prices and hence international exchange rates, in response to shocks. This persistence channel is aided by the second mechanism in our model: capital complementarity. Capital complementarity provides for an additional return on capital which places additional restriction on the equilibrium asset pricing relations for money and capital. We also show in section 4.1 how this may further introduce persistence in equilibrium relative prices and international exchange rates.

In our model, the assumption of (some) anonymous trades in the DM is intertwined with the DM as a non-traded goods sector. To disentangle the contribution of anonymity and the role of the non-tradable sector on the exchange rate dynamics, we relax the anonymity assumption, as in Aruoba, Waller, and Wright [2009]. In particular, we introduce an exogenous probability that agents in each DM may be segmented into one of two kinds of trades: anonymous monetary trades or monitored credit trades. By considering the limit of pure credit trades in the DM, we are able to shut down the role of monetary friction and isolate the effect on exchange rate dynamics due to the non-monetary DM as a non-traded goods sector. We show that the latter feature alone cannot account for the stylized facts on the exchange rates for the U.S. However, in the presence of a small degree of anonymity in the DM, cross-country aggregate relative prices are non-volatile and persistent, in response to aggregate technology and money supply growth shocks. This contributes to the excess volatility and persistence in the real and nominal exchange rates. Thus, without requiring exogenous price-stickiness [e.g. Chari, Kehoe, and McGrattan, 2002] nor additional shocks [e.g. Steinsson, 2008], the benchmark model is also able to rationalize near perfect positive correlation between the real and nominal exchange rate. To be sure that the second mechanism of capital complementarity is not a key driver of the results, we also consider the limit where this complementarity is not present. Again, we show that the real exchange rate exhibits the stylized fact of excess volatility only when there is a monetary or information friction. Thus the monetary or information friction, in the sense of Lagos and Wright [2005], is more than just a vehicle for a theoretical foundation of money. In a stochastic two-country environment, it restricts pricing relations such that the model is able to account for the stylized facts on real and nominal exchange rate fluctuations.

The paper is organized as follows. In section 2, we outline the details and assumption of the baseline quantitative-theoretical model. We then work through the model's stationary Markov monetary equilibrium and its implications in Section 3. Next, in Section 4, we provide some insight into the key mechanisms in the model, and explain the potential trade-offs and the role of the DM pricing protocol in accounting for relative pricing and exchange rate behavior. We then take the theory to the data in Section 5. We discuss the model's business cycle features relative to the data and other existing models in Section 6. We then verify how the mechanisms interact to produce the business cycle features, by isolating each mechanism, in Section 6.1. We conclude in Section 7.

2. ENVIRONMENT

Consider a two-country model, each referred to as Home and Foreign. Variables and parameters without an asterisk (or with a subscript h) will refer to the Home country, and those with an asterisk (or with a subscript f), will refer to the Foreign country. Time is denumerable, and a time period is denoted by $t \in \mathbb{N} := \{0, 1, 2, ...\}$. Agents exist on a continuum $[0, 1]$ and have a common discount factor $\beta \in (0, 1)$. Each $t \in \mathbb{N}$ is composed of two arbitrary sub-periods, night and day. At night, agents trade anonymously in decentralized markets (DM). During the day, agents trade in Walrasian centralized markets (CM). The nature of consumption, production and trade in each market will be explained in detail in sections 2.6 and 2.7.

2.1. **Preferences and DM technology.** Denote $q^b \in \mathbb{R}_+$ as an agent's consumption (as a buyer) and $q^s \in \mathbb{R}_+$ as an agent's output (as a seller) of a "specialized", or, agent-specific and non-storable good in the DM. Similar to Lagos and Wright [2005], each agent can be a producer of a special q^s, and is assumed to not value his own product. Let $X \in \mathbb{R}_+$, $k \in \mathbb{R}_+$ and $H \in [0, \overline{H}]$, where $\overline{H} < +\infty$, denote consumption of a general good in the CM, individual capital stock and labor in the CM, respectively. Agents' per-period preferences are represented by $(q^b, q^s, X, H, z) \mapsto u(q^b) - c(q^s/z, k) + U(X) - h(H)$, where $u(q)$ is the per-period payoff from consuming q, z is aggregate home total factor productivity, and $c(q/z, k)$ is the utility cost of producing q with fixed within-period capital, k, determined in the previous CM.[2] $U(X)$ is the immediate payoff from consuming X in the CM, and $-h(H)$ is the disutility of work effort in the CM. We make the following assumptions.

Assumption 1. *The functions $u, U, h : \mathbb{R}_+ \to \mathbb{R}$ and $c : \mathbb{R}^2_+ \to \mathbb{R}$ have the following properties:*

(i) *First and second derivatives exist everywhere: $u, U \in \mathbf{C}^2(\mathbb{R}_+)$ and $c \in \mathbf{C}^2(\mathbb{R}^2_+)$;*
(ii) $u_q > 0$, $c_q > 0$, $c_k < 0$, $U_X > 0$, $h_H > 0$ *and constant;*
(iii) $u_{qq} < 0$, $c_{qq} \geq 0$, $c_{qk} < 0$, $U_{XX} \leq 0$ *and* $h_{HH} = 0$*;*
(iv) $u(0) = c(0, 0) = 0$*; and*
(v) $u(q) > c(q/z, k)$ *for every* $(q/z, k)$.

2.2. **DM access (or matching) technology.** In our benchmark economy with DM competitive price taking, we assume that there is a probability $\sigma \leq 1/2$ that each agent can access the DM as a buyer. With symmetric probability σ, the agent can access the DM to sell his special good. With probability $1 - 2\sigma$, an agent cannot access the DM, or equivalently, will leave the DM with no exchange.[3] For simplicity, assume that "double-coincidence-of-wants" events (where buyers and sellers in the DM

[2] Or equivalently, let H_{DM} be the labor effort of an agent expended in a DM. Suppose the production technology, $(H_{DM}, k, z) \mapsto z \cdot \tilde{F}(H_{DM}, k)$ using capital and labor, is bijective and homogeneous of degree one. Then $H_{DM} = \tilde{F}^{-1}(q^s/zk) \cdot k$ and $c(q^s/z, k) \equiv H_{DM}$. Our quantitative exercise will use a Cobb-Douglas example for $\tilde{F}(\cdot, \cdot; \varpi)$ where $1/\varpi \in (0, 1)$ is the labor share.

[3] As pointed out by Rocheteau and Wright [2005], this "competitive equilibrium" interpretation can be thought as a generalization of Lucas and Prescott [1974] and Alvarez and Veracierto [2000] and is still consistent with the essentiality of money, as long as we maintain anonymity and events with a double-coincidence-of-wants problem. Later on, when we consider DM bargaining (proportional and generalized Nash bargaining) in bilateral matches, the interpretation of σ then is that of either the probability that the agent as buyer meets a seller of a special good he wishes to consume, or, the symmetric probability that the same agent, as seller, meets a buyer who wants to buy his product.

are able to barter), and, the event where an agent can simultaneously buy q^b and sell q^s, occur with probability zero.

2.3. **CM technology.** In the CM the final good in the Home country is produced according to a constant returns technology, $(y_h, y_f) \mapsto G(y_h, y_f)$, where y_h denotes the input demand for an intermediate good produced in the home country, and, y_f represents the demand of a substitutable input produced in the foreign country. Assume that $G \in \mathbf{C}^2(\mathbb{R}^2_+)$, $G_i > 0, G_j > 0, G_{ii} < 0$, and $G_{jj} < 0$, where $i, j \in \{y_h, y_f\}$. Similarly, the foreign final good production function is given by, $(y^*_f, y^*_h) \mapsto G(y^*_f, y^*_h)$.

Let K denote an aggregate capital stock in each home country. The production of the different intermediate goods are given by another constant returns technology, $(K, H) \mapsto zF(K, H)$ which is subject to a stochastic productivity shock, z. Assume $(z_t)_{t\in\mathbb{N}}$ is a strictly positive and bounded stochastic process. Assume that $F \in \mathbf{C}^2(\mathbb{R}^2_+)$ and that $F_K > 0, F_H > 0, F_{KK} < 0, F_{HH} < 0$, and, $F(K, 0) = F(0, H) = 0$.

2.4. **State variables.** Let $m \in \mathbb{R}_+$ be the stock of an agent's local nominal money holding in the Home country.[4] Denote b as the current stock of an internationally traded complete state-contingent money claim, held by an agent in the Home country. Each b is denominated in the Home currency. Since these complete contingent claims require knowledge of traders' histories, it is natural that they are not issued or traded in the DM with anonymous randomly matched trades. They are traded only during each CM subperiod. We assume that k cannot be used as a means of payment in the DM since it is not portable.[5]

Now we introduce a modelling device that will help us identify the role of anonymity or monetary friction in the model. Following Aruoba, Waller, and Wright [2009], suppose that conditional on the events of buying, or selling, the exogenous probability that a buyer or seller would engage in an exchange where record keeping is possible is $(1 - \kappa) \in [0, 1]$. That is, the event that a buyer *or* a seller can buy or sell a good in the DM using credit occurs with the discrete probability measure $\sigma(1 - \kappa)$. Since credit is assumed to be enforceable in such an event, a buyer is willing to take (and a seller is willing to give) out the nominal loan l in exchange for a good, say $\check{q}$. This loan is

[4] Given that some medium of exchange is essential in the DM, one issue in monetary theory is to determine endogenously which objects serve this function. This issue becomes more involved when there are multiple currencies in an international setting. In our model, we consider the restriction that agents can only use the local currency to buy local goods both in DM and CM. One possible explanation rationalizing this restriction lies in sellers' unwillingness to accept a foreign currency or assets that they do not recognize as a result of private information. These underlying private information problems in payment arrangements are examined more closely by Lester, Postlewaite, and Wright [2008] and Li and Rocheteau [2009]. However, these explorations are beyond the scope of this paper.

[5] In the DM our agents have their capital physically fixed in place at production sites. Thus, a buyer must visit randomly the location of a seller, and since capital is not portable, it cannot be used for payment, while currency can. This use of spatial separation is in the spirit of the "worker-shopper" idea.

required to be repaid in full in the following CM. Then we let q denote a DM specialized good that is exchanged for money in events where exchange occurs with measure $\sigma\kappa$ for a buyer *or* seller.

Thus we have two distinct markets, one for anonymous traders where cash is needed and one where credit is available. In particular, a fraction $\sigma(1-\kappa)$ of agents can trade in DM with credit, while a fraction $\sigma\kappa$ of agents trade only using fiat money. This is useful because when $\kappa = 0$, we are able to shut down the source of monetary friction – the anonymity assumption – and the resulting limit economy is a version of a two-sector real business cycle model with traded and nontraded goods.

Denote the vector of exogenous shocks as $\mathbf{z} \in Z$. We consider Home and Foreign, technology (z) and money supply growth (ψ), shocks. Thus $\mathbf{z} := (z, z^*, \psi, \psi^*)$, and Z is a compact cube in $\mathbb{R}^2_+ \times \mathbb{R}^2$. Let the time-$t$ *aggregate* (global) CM state vector relevant to an agent in country $i \in \{h, f\}$ be $\mathbf{s} := (M, M^*, B, B^*, K, K^*, \phi, \phi^*, e, \mu_h, \mu_f, \mathbf{z})$. These state variables are defined as follows. The Home aggregate money stock, total private state contingent claims, and capital stock are, respectively, M, B and K. The value of money in the Home CM is $\phi := 1/p_X$, where p_X is the price level of the Home CM general goods. Similarly, the asterisked variables pertain to the Foreign country's aggregate state variables. The nominal exchange rate in Home CM currency terms is e. For country i, $\mu_i(\cdot, \mathbf{z}) : \mathcal{B}_i(\mathbf{z}) \to [0, 1]$ is the time-t probability measure on the Borel σ-field $\mathcal{B}_i(\mathbf{z})$ generated by (m, b, k, l), at each vector of exogenous state variables, $\mathbf{z}$.[6]

At the beginning of the time-t DM, the *aggregate* (global) state vector for an agent in country $i \in \{h, f\}$ is $\hat{\mathbf{s}} := (M, M^*, B, B^*, K, K^*, \phi, \phi^*, e, \nu_h, \nu_f, \mathbf{z})$. The explicit switch in notation from ν_i to μ_i takes into account that, in general, the distribution of assets upon the economy i entering each period's DM, ν_i, may be different to the distribution μ_i upon its leaving the DM, and into the CM, in the same period.[7]

2.5. **Timing.** Figure 1 depicts the sequence of events within each $t \in \mathbb{N}$. The relevant aggregate state vector $\mathbf{s}$ is realized at the beginning of each t. This is public information for all agents. An agent in the Home country, first entering the DM with assets $(m, b, k, l) = (m, b, k, 0)$, given $\hat{\mathbf{s}}$, is publicly known by the *individual* state $(\mathbf{a}, \hat{\mathbf{s}}) := (m, b, k, 0, \hat{\mathbf{s}})$. His indirect utility value of that state is $V(\mathbf{a}, \hat{\mathbf{s}})$. For simplicity, we make the restriction that each country-i agent does not hold another country's currency as an asset.[8] Since trading opportunities in the DM are random, agents within each country i only know the state of their trade partners *ex post*. *Ex ante* they

[6]Note that if $Z = \emptyset$, *i.e.* in the absence of aggregate exogenous shocks, then the solution of the Markov equilibrium is characterized by a deterministic difference equation system, as in Lagos and Wright [2005]. Also, note that the aggregate prices (ϕ, ϕ^*, e) are explicitly included as (auxiliary) state variables, following Duffie, Geanakoplos, Mas-Colell, and McLennan [1994], so that we can restrict our characterization of equilibria to stationary Markov equilibria.

[7]It is straightforward to prove that the probability measures ν_i for each $i \in \{h, f\}$, is degenerate in any equilibrium, as a stochastic extension to the original proof in Lagos and Wright [2004]. This affords us plenty of tractability and ease of computation later.

[8]See Head and Shi [2003] for the environment where agents trade currency internationally.

only know the probability distribution of traders in the DM, which is $(\sigma, \sigma, 1-2\sigma)$ with support $\{Buyer, Seller, Neither\}$. Conditional on either events $\{Buyer\}$ or $\{Seller\}$, there is an identical distribution $\{\kappa, 1-\kappa\}$ faced by the agent of a trade being either anonymous (monetary) or monitored (credit).

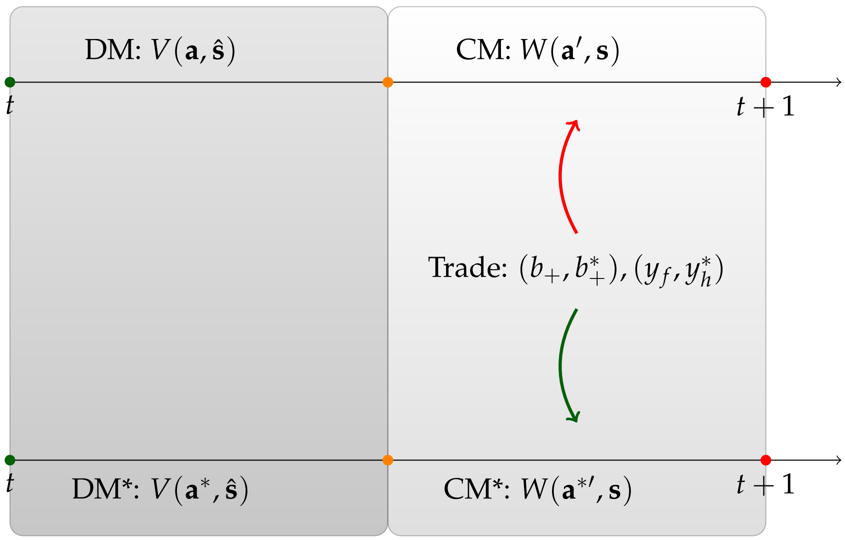


FIGURE 1. Timing

Upon leaving the DM, an agent's *individual* state changes to

$$(\mathbf{a}', \mathbf{s}) := \begin{cases} (m', b, k, 0, \mathbf{s}) & \text{w.p. } 2\sigma\kappa \\ (m, b, k, l, \mathbf{s}) & \text{w.p. } 2\sigma(1-\kappa) \end{cases}$$

reflecting the possibility that money had changed hands as a result of the agent being a buyer or seller. As a result of that, the distribution of assets (namely money) would also have changed from $\nu_i \in \hat{\mathbf{s}}$ to $\mu_i \in \mathbf{s}$. The components (b, k) have not changed since they are predetermined at the beginning of t. Thus, within t, the agent enters the CM with possible state $(\mathbf{a}', \mathbf{s})$ and his value of that state is $W(\mathbf{a}', \mathbf{s})$. Agents do not discount payoffs within each period t.

In the next two sections we describe in detail the sub-period problems, DM and CM, in a backward fashion. To economize on notation, we use the following convention. A variable or vector with a "$+$" subscript will denote its time $t+1$ contingent outcome. A state with a "$-$" subscript will denote its time $t-1$ realization. However, in some cases, variables with a "$+$" subscript, such as money, capital and bonds, are predetermined at the beginning of time $t+1$. In such cases, these are decision or control variables which will be made obvious in the problems below. The same variable without the "$+$" or "$-$" subscript denotes its current or time-t realization.

2.6. **Centralized markets.** In the Home CM, an agent consumes a general good $X \in \mathbb{R}_+$ which is produced using CM-specific labor $H \in \mathbb{R}_+$ and capital k. In contrast to Lagos and Wright [2005], we introduce a set of internationally traded complete nominal state-contingent claims. Agents in each country's CM who consume more (less) than their total wealth can also trade in these securities.

Let $h(H) = A \cdot H$, where $A > 0$ is a constant marginal disutility of work effort. Let $\delta \in [0,1]$ be the depreciation rate of capital and τ_K a proportional tax rate on capital income. Denote $\tilde{r}(\mathbf{s})$ and $\tilde{w}(\mathbf{s})$ as competitive rates of return to capital and labor services, respectively. Then $r := r(\mathbf{s}) \equiv (1 - \tau_K)(\tilde{r}(\mathbf{s}) - \delta)$ is the after-tax rate of return to capital, net of depreciation. Similarly, $w(\mathbf{s}) := (1 - \tau_H)\tilde{w}(\mathbf{s})$ is the after-tax real wage rate. Denote τ_X as the proportional tax rate on CM consumption X. Let $m_+ := m(\mathbf{a}, \mathbf{s})$, $k_+ := k(\mathbf{a}, \mathbf{s})$, and $b_+ := b(\mathbf{a}, \mathbf{s})$, so that $\mathbf{a}_+ = (m_+, b_+, k_+, 0)$. $Q(\mathbf{a}_+, \mathbf{s}_+ | \mathbf{a}, \mathbf{s})$ is the domestic price of one unit of the state-contingent claim $b(\mathbf{a}_+, \mathbf{s}_+ | \mathbf{a}, \mathbf{s})$. Let $\phi := \phi(\mathbf{s}) = 1/p_X(\mathbf{s})$ be the inverse of the price of X (*i.e.* the CM-good value of a unit of Home currency) in the Home country.

At each $t \in \mathbb{N}$, a price-taking agent (at the beginning of the CM sub-period in the Home country) named $(m, b, k, l, \mathbf{s})$ solves the recursive problem given by

$$W(m, b, k, l, \mathbf{s}) = \max_{X, H, m_+, k_+, b_+} \left\{ U(X) - AH + \beta \int V(m_+, b_+, k_+, 0, \mathbf{s}_+) \lambda(\mathbf{s}, d\hat{\mathbf{s}}_+) \right\} \tag{1}$$

subject to

$$\mathbf{s}_+ = \mathcal{G}(\mathbf{s}, \mathbf{v}_+), \qquad \mathbf{v} \overset{\text{i.i.d.}}{\sim} \varphi, \tag{2}$$

and,

$$\begin{aligned} &(1 + \tau_X) X(\mathbf{a}, \mathbf{s}) + k(\mathbf{a}, \mathbf{s}) - k - \phi(\mathbf{s}) b + T(\mathbf{s}) \\ &\qquad = \phi(\mathbf{s}) \left[m - m(\mathbf{a}, \mathbf{s}) - l \right] + w(\mathbf{s}) H(\mathbf{a}, \mathbf{s}) + r(\mathbf{s}) k \\ &\qquad - \phi(\mathbf{s}) \iint_{\mathbf{s}_+, \mathbf{a}_+} b(\mathbf{a}_+, \mathbf{s}_+ | \mathbf{a}, \mathbf{s}) Q(\mathbf{a}_+, \mathbf{s}_+ | \mathbf{a}, \mathbf{s}) \mu_h(\mathbf{s}_+, d\mathbf{a}_+) \lambda(\mathbf{s}, d\mathbf{s}_+), \end{aligned} \tag{3}$$

where $\lambda(\mathbf{s}, \cdot)$, for each given $\mathbf{s}$, is induced by $\mathcal{G} \circ \varphi$, and defines an equilibrium product probability measure over Borel-subsets containing $\hat{\mathbf{s}}_+$. Constraint (2) describes a transition law, where the mapping $\mathcal{G} = \mathcal{G}_{\{\mathbf{s}\} \setminus \{\mathbf{z}\}} \circ \mathcal{G}_{\{\mathbf{z}\}}$, with component $\mathcal{G}_{\{\mathbf{s}\} \setminus \{\mathbf{z}\}}$ inducing the $\mathbf{z}$-dependent stochastic process for endogenous aggregate states, $\{\mathbf{s}\} \setminus \{\mathbf{z}\}$, is to be pinned down in equilibrium, and $(\mathbf{z}, \mathbf{v}_+) \mapsto \mathcal{G}_{\{\mathbf{z}\}}(\mathbf{z}, \mathbf{v}_+)$ is an exogenous map for the aggregate shocks. Implicit in constraint (2) is the equilibrium transition of the distribution of *individual* states from the period-t CM, to the period-$(t+1)$ DM,

$$\nu_h(\hat{\mathbf{s}}_+, \cdot) = \mathcal{G}_\nu \left[\mu_h(\mathbf{s}, \cdot), \mathbf{z}_+ \right], \tag{4}$$

such that the relevant conditional distribution of assets at the beginning of the time-$(t+1)$ CM subperiod is given by

$$\mu_h(\mathbf{s}_+, \cdot) = \mathcal{G}_\mu [\nu_h(\hat{\mathbf{s}}_+, \cdot), \mathbf{z}_+] \equiv \mathcal{G}_\mu \circ \mathcal{G}_\nu(\mathbf{s}, \mathbf{z}_+), \tag{5}$$

where $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ are components of $\mathcal{G}_{\{\mathbf{s}\}\setminus\{\mathbf{z}\}}$.

The sequential state-contingent one-period budget constraint given by (3) says the following. For each given state $(m, b, k, l, \mathbf{s})$, taxable consumption of the general good X is to be financed by the change in real money holdings, by after-tax real labor income wH, after-tax real capital income rk, net of investment flows to physical capital made in the CM, net of contingent claims in real terms, and net of lump-sum government taxes, T.

2.6.1. *Optimal individuals' decisions in the CM.* Eliminating H in (1), using the budget constraint (3), the optimal decision rules satisfy the following conditions for every state $(\mathbf{a}, \mathbf{s})$ and every measurable event containing the continuation state $(\mathbf{a}_+, \hat{\mathbf{s}}_+)$.

The optimal trade-off between current CM consumption X and leisure $-H$, given the after-tax real wage $w := w(\mathbf{s})$, is

$$X: \qquad U_X [X(\mathbf{a}, \mathbf{s})] = \frac{A(1+\tau_X)}{w(\mathbf{s})}. \tag{6}$$

The optimal trade-off between a current increase in marginal utility of X in the CM and the present-value expected marginal value of entering the next-period DM with a marginal increment of money holdings is

$$m_+: \qquad \frac{A\phi(\mathbf{s})}{w(\mathbf{s})} = \beta \int V_{m_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+) \lambda(\mathbf{s}, d\hat{\mathbf{s}}_+). \tag{7}$$

Similar to condition (7), conditions (8)-(9) below provide the optimal trade-offs between the current utility of consumption of X and the expected discounted marginal value of entering the DM with more assets. Specifically, the optimal choice of the complete state-contingent money claims, or bonds, is given by

$$b_+(\cdot; \mathbf{s}): \qquad \frac{A\phi(\mathbf{s})}{w(\mathbf{s})} [Q(\mathbf{a}_+, \mathbf{s}_+ | \mathbf{a}, \mathbf{s}) \mu_h(\mathbf{s}_+, d\mathbf{a}_+)] \lambda(\mathbf{s}, d\hat{\mathbf{s}}_+) = \beta V_{b_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+), \tag{8}$$

which holds for every $\mathbf{s}$, every $\hat{\mathbf{s}}_+$, and implicitly, every $\mathbf{s}_+$.

The optimal choice of the Home-produced capital stock available for production in the next period satisfies

$$k_+: \qquad \frac{A}{w(\mathbf{s})} = \beta \int V_{k_+}(m_+, b_+, k_+, 0, \hat{\mathbf{s}}_+) \lambda(\mathbf{s}, d\hat{\mathbf{s}}_+). \tag{9}$$

2.6.2. *Envelope conditions in the CM.* At an optimum, the envelope conditions for the agent's CM decision problem are as follows. The marginal value of money holdings upon entering the CM is

$$W_m(m,b,k,l,\mathbf{s}) = \frac{A\phi(\mathbf{s})}{w(\mathbf{s})}, \tag{10}$$

the marginal value of holding bonds upon entering the CM, respectively, are

$$W_b(m,b,k,l,\mathbf{s}) = \frac{A\phi(\mathbf{s})}{w(\mathbf{s})}, \tag{11}$$

and the marginal value of holding the each of the four types of capital stocks at the beginning of the CM are as follows. With respect to a Home agent's holding of capital stock in the Home country, the marginal CM value is

$$W_k(m,b,k,l,\mathbf{s}) = \frac{A}{w(\mathbf{s})}\left[1+r(\mathbf{s})\right]. \tag{12}$$

With respect to a Home agent's holding of credit in the Home country, the marginal CM value is

$$W_l(m,b,k,l,\mathbf{s}) = -\frac{A\phi(\mathbf{s})}{w(\mathbf{s})}. \tag{13}$$

The envelope conditions (10)-(13) imply that, W is linear in (m,b,k,l), for each fixed aggregate state $\mathbf{s}$. So we can write W as

$$W(m,b,k,l,\mathbf{s}) = W(0,0,0,0,\mathbf{s}) + \frac{A}{w}\Big[\phi(m+b)+(1+r)k\Big]. \tag{14}$$

2.6.3. *Firms.* Let P_h be the Home currency price of the Home produced intermediate good, and P_y be that of the Foreign produced intermediate good use by the Home final-good firm. The Home final-good firm solves

$$\max_{y_h,y_f}\left\{\frac{G[y_h(\mathbf{s}),y_f(\mathbf{s})]}{\phi(\mathbf{s})} - P_h(\mathbf{s})y_h(\mathbf{s}) - P_f(\mathbf{s})y_f(\mathbf{s})\right\}.$$

The profit-maximizing conditions are:

$$\phi(\mathbf{s})P_h(\mathbf{s}) = G_{y_h}[y_h(\mathbf{s}),y_f(\mathbf{s})], \tag{15}$$

and

$$\phi(\mathbf{s})P_f(\mathbf{s}) = G_{y_f}[y_h(\mathbf{s}),y_f(\mathbf{s})]. \tag{16}$$

The Home intermediate goods producer solves

$$\max_{H,K}\left\{P_{y_h}(\mathbf{s})\cdot zF_k[K(\mathbf{s}_-),H(\mathbf{s})] - \frac{[\tilde{w}(\mathbf{s})H(\mathbf{s})+\tilde{r}(\mathbf{s})K(\mathbf{s}_-)]}{\phi(\mathbf{s})}\right\}.$$

where the market for inputs to F is perfectly competitive. Profit maximization is characterized by the usual first order conditions where capital and labor are paid a respective rental rate which equals their marginal products in every aggregate state $\mathbf{s}$:

$$\tilde{r}(\mathbf{s}) = \phi(\mathbf{s})P_h(\mathbf{s}) \cdot zF_k[K(\mathbf{s}_-), H(\mathbf{s})], \tag{17}$$

and

$$\tilde{w}(\mathbf{s}) = \phi(\mathbf{s})P_h(\mathbf{s}) \cdot zF_H[K(\mathbf{s}_-), H(\mathbf{s})], \tag{18}$$

where

$$H(\mathbf{s}) = \int_{\mathbf{a}} H(\mathbf{a}, \mathbf{s})\mu_h(\mathbf{s}, d\mathbf{a})$$

is aggregate labor supply in the Home CM.

A foreign country's CM agent named $(m^*, b^*, k^*, l^*, \mathbf{s})$ and its firm have a symmetric problem to (1)-(3), (15)-(16), and (17)-(18).

2.7. **Decentralized markets.** At the beginning of each $t \in \mathbb{N}$, an agent named $(m, b, k, 0, \hat{\mathbf{s}})$ enters the DM.[9] With a fixed probability σ this agent is the buyer of the special good that some other agent produces, q^b, where the other agent (seller) is indexed by the state $(\tilde{\mathbf{a}}, \hat{\mathbf{s}}) := (\tilde{m}, \tilde{b}, \tilde{k}, 0, \hat{\mathbf{s}})$, but not vice-versa. With probability $\sigma\kappa$, the buyer parts with d^b "dollars" and realizes a payoff of $u(q^b) \in \mathbb{R}$. The buyer then enters the day CM with a value of $W\left(m - d^b, b, k, 0, \mathbf{s}\right)$. With probability $\sigma(1-\kappa)$, the buyer does not use money, but takes out a nominal loan l, from the seller he meets, and realizes a payoff of $u(\breve{q}^b) \in \mathbb{R}$. The buyer then enters the day CM with a value of $W(m, b, k, l, \mathbf{s})$.

Symmetrically, with probability $\sigma\kappa$, agent $(m, b, k, 0, \hat{\mathbf{s}})$ has a special good q^s which other buyers want to buy, but not vice-versa. This agent receives d^s dollars in exchange for exerting a utility cost of production $c(q^s/z, k) \in \mathbb{R}_+$. Notice that capital obtained from the previous period's CM, k, accrues a return in the DM in the form of the marginal benefit to producing q (q^s or $\breve{q}^s$), i.e. $c_k(q/z, k)$.[10] This seller then enters the day CM with a value of $W(m + d^s, b, k, 0, \mathbf{s})$. With probability $\sigma(1-\kappa)$, a seller may sell $\breve{q}^s$ by extending a loan l to a matched buyer.

These four events described above are known as single-coincidence-of-wants meetings, where money is a portable medium of exchange in events that occur with probability $2\sigma\kappa$, and where credit l is the medium of exchange in events with probability $2\sigma(1-\kappa)$. With probability $1-2\sigma$, agent $(m, b, k, 0, \hat{\mathbf{s}})$ leaves the DM and enters the day with his assets intact, and begins his activity in the CM with value $W(m, b, k, 0, \mathbf{s})$. For simplicity, we assume the probability of a "double-coincidence" meeting, and hence the occurrence of pure barter, is zero.

[9]Note that m implicitly includes any aggregate monetary transfer or injection from the government, which we denote later as $\iota(\hat{\mathbf{s}})$, so then, $m(\hat{\mathbf{s}}) = m(\mathbf{s}_-) + \iota(\hat{\mathbf{s}})$.

[10]This feature was first introduced by Aruoba, Waller, and Wright [2009, Appendix A.1]. The authors showed that whether there exist two kinds of capital goods, for use in the DM and in the CM production, respectively, is of negligible quantitative consequence in their model.

Formally, an agent named $(m, b, k, 0, \hat{\mathbf{s}})$ has a value $V(m, b, k, 0, \hat{\mathbf{s}})$ at the beginning of the DM that satisfies the following problem:

$$\begin{aligned} V(m, b, k, 0, \hat{\mathbf{s}}) = &\sigma V^b(m, b, k, 0, \hat{\mathbf{s}}) \\ &+ \sigma V^s(m, b, k, 0, \hat{\mathbf{s}}) + (1 - 2\sigma) W(m, b, k, \mathbf{s}). \end{aligned} \tag{19}$$

where, in general:

$$\begin{aligned} V^b(m, b, k, 0, \hat{\mathbf{s}}) =&\kappa \int \left[u(q^b) + W\left(m - d^b, b, k, 0, \mathbf{s} \right) \right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}) \\ &+ (1 - \kappa) \int \left[u(\breve{q}^b) + W\left(m, b, k, l^b, \mathbf{s} \right) \right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}), \end{aligned}$$

and,

$$\begin{aligned} V^s(m, b, k, 0, \hat{\mathbf{s}}) =&\kappa \int \left[-c(q^s, k) + W\left(m + d^s, b, k, 0, \mathbf{s} \right) \right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}) \\ &+ (1 - \kappa) \int \left[-c(\breve{q}^s, k) + W\left(m, b, k, -l^s, \mathbf{s} \right) \right] \nu_h(d\tilde{\mathbf{a}}, \hat{\mathbf{s}}). \end{aligned}$$

are the value functions of ex-post buyer and sellers respectively.

2.7.1. *Walrasian price taking.* Consider a version of the DM where $(q^b, q^s, \tilde{p}, \breve{p}, \breve{q}^b, \breve{q}^s, l^b, l^s)$ are determined by Walrasian price taking. Then, we have

$$\begin{aligned} V^b(m, b, k, 0, \hat{\mathbf{s}}) = &\kappa \max_{q^b \in [0, m/\tilde{p}]} \left[u(q^b) + W\left(m - \tilde{p} q^b, b, k, 0, \mathbf{s} \right) \right] \\ &+ (1 - \kappa) \max_{\breve{q} \in [0, l^b/\breve{p}]} \left[u(\breve{q}^b) + W\left(m, b, k, l^b, \mathbf{s} \right) \right], \end{aligned}$$

where $d^b = \tilde{p} q^b$, and,

$$\begin{aligned} V^s(m, b, k, 0, \hat{\mathbf{s}}) = &\kappa \max_{q^s} \left[-c(q^s/z, k) + W\left(m + \tilde{p} q^s, b, k, 0, \mathbf{s} \right) \right] \\ &+ (1 - \kappa) \max_{\breve{q}^s} \left[-c(\breve{q}^s/z, k) + W\left(m, b, k, -l^s, \mathbf{s} \right) \right], \end{aligned}$$

where $d^s = \tilde{p} q^s$, $\tilde{p}$ and $\breve{p}$ are the respective prices of a special good in anonymous and monitored trades, taken as given by all buyers and sellers.

2.8. **Government.** New money is injected at the end of the period in the CM.[11] Specifically, the monetary authority follows a monetary supply rule:

$$M(\mathbf{s}) = \exp(\psi) M(\mathbf{s}_-), \tag{20}$$

[11] This is merely for mathematical convenience, so that within each DM, agents do not have to deal with a stochastic total payoff function, W.

where $\exp\{\psi\} - 1$ is the one-period money supply growth rate between time t and $t+1$. Assume that $(\exp(\psi_t))_{t\in\mathbb{N}}$ follows a Markov process that lives in the compact set $[1, N]$, with $N < +\infty$. We define this process later.

Government expenditure G^d is financed by lump-sum taxes/transfers, seigniorage and consumption, labor and capital tax revenue:

$$G^d(\mathbf{s}) = [T(\mathbf{s}) + (M(\mathbf{s}) - M(\mathbf{s}_-))\phi(\mathbf{s})] + \tau_X X(\mathbf{s}) + \tau_H H(\mathbf{s}) + \tau_K(\tilde{r}(\mathbf{s}) - \delta)K(\mathbf{s}_-). \tag{21}$$

We assume that $T(\mathbf{s}) = -(M(\mathbf{s}) - M(\mathbf{s}_-))\phi(\mathbf{s})$.

3. Stationary Markov Monetary Equilibrium

In this section, we state a key result which is just an extension of Lagos and Wright [2005] to environments with aggregate uncertainty.[12] In an equilibrium, the endogenous distribution of agents' asset holdings is degenerate at the start of each period (and hence DM), such that all agents in each country choose the same allocations that depend only on the global state. We further characterize the equilibrium conditions in the DM and list the conditions for market clearing in the CM. We then define the elements that constitute a *stationary Markov monetary equilibrium*.

In general, because of the random meeting technology in the DM, we will need to track the history of aggregate distribution of assets held by agents in any equilibrium where money has value. However, because of the quasi-linear assumption on each agent's per-period payoff function, it can be shown that in equilibrium asset holdings at the beginning of each $t \in \mathbb{N}$ are identical across all agents within each country i, so that,

$$\begin{aligned}(m, b, k, 0)(\mathbf{s}) &= \int (m, b, k, 0)\nu_i(\hat{\mathbf{s}}, dm, db, dk, dl) \\ &:=: (M, B, K, 0)(\hat{\mathbf{s}}) \\ &=: (M, B, K, 0)(\mathbf{z}).\end{aligned} \tag{22}$$

for each $i \in \{h, f\}$, for all $\hat{\mathbf{s}}$. This implies that we can explicitly write $\nu(\hat{\mathbf{s}}, \cdot)$ as $\nu(\mathbf{z}, \cdot)$, and furthermore, for every $\mathbf{z}$, and every $A \in \mathcal{B}_i(\mathbf{z})$,

$$\nu_i(\mathbf{z}, A) = \begin{cases} 1 & \text{if } (m, b, k, 0) = (M, B, K, 0) \in A \\ 0 & \text{otherwise} \end{cases}.$$

However, we can see that even if $\nu_i(\mathbf{z}, \cdot)$ is degenerate at the end of the CM, $\mu_i(\mathbf{z}, \cdot)$ is not. Thus, explicitly, agents at the beginning of each CM will still face an aggregate state variable $\mathbf{s}$ that contains a non-degenerate distribution of individual states. Specifically, the non-degeneracy is along the dimension of money holdings out of the DM.

[12] A proof is available upon request from the authors.

3.1. **DM competitive pricing and equilibrium decisions.** In equilibrium, the constraints $d \leq m$, and $l \leq \check{p}\check{q}$ bind, and $q^b = q^s = q$. Thus for the $\sigma\kappa$ proportion of agents who are sellers that meet buyers and they trade with money, we have the equilibrium condition that the marginal utility value to the buyer of a unit of the home currency (for buying q), is equal to the marginal utility cost of production of the DM seller:

$$\frac{A\phi}{w}M = \frac{1}{z}c_q(q/z,K)q \equiv g(q,K,z). \tag{23}$$

Note that $\tilde{p} = M/q$ in equilibrium. If we assume alternative DM protocols for determining the terms of trade – *e.g.* generalized Nash bargaining – then the function g, which would represent a bilateral buyer-seller sharing function, will be quite different.[13]

For the $\sigma(1-\kappa)$ proportion of buyers and sellers, we have:

$$\frac{A\phi}{w}l = \frac{1}{z}c_q(\check{q}/z,K)\check{q} \equiv g(\check{q},K,z). \tag{24}$$

Since by assumption contracts are enforceable for these agents, then credit attains the first best DM allocation in terms of $\check{q}$ satisfying

$$u_q(\check{q}) = \frac{1}{z}c_q(\check{q}/z,K). \tag{25}$$

Therefore we can substitute out credit in the equilibrium conditions later, using

$$l = \frac{wu_q(\check{q})\check{q}}{A\phi}. \tag{26}$$

3.2. **Envelope conditions in the DM.** At an interior optimum consistent with equilibrium, we have the following envelope conditions. Utilizing the linearity of W, the marginal value of money at the beginning of the DM is

$$V_M(M,B,K,0,\hat{\mathbf{s}}) = \frac{A\phi}{w}\left[(1-\sigma\kappa) + \sigma\kappa\frac{z \cdot u_q(q)}{c_q(q/z,K)}\right] > 0. \tag{27}$$

The marginal value of the state-contingent money claims at the beginning of the DM is

$$V_B(M,B,K,0,\hat{\mathbf{s}}) = W_b(M,B,K,0,\mathbf{s}) = \frac{A\phi}{w}. \tag{28}$$

The DM marginal value of the capital stock, is

$$V_K(M,B,K,0,\hat{\mathbf{s}}) = \frac{A\phi}{w}(1+r) - \sigma\kappa\gamma(q,K,z) - \sigma(1-\kappa)\gamma(\check{q},K,z) > 0, \tag{29}$$

[13] These alternatives are considered quantitatively later, and discussed in detail in a separate Appendix available upon request.

where

$$\gamma(q, K, z) = c_K(q/z, K) < 0. \tag{30}$$

The function γ is strictly negative due to two effects that capture the reduction in marginal cost of production in the DM. The first term on the right of (30) is the indirect effect on marginal cost through the effect of an additional capital stock on the terms of trade q.

3.3. **Market clearing in the CM.** In an equilibrium, since agents within each country choose the same asset holdings, i.e. $(m, b, k) = (M, B, K)$, then they do not borrow from, or, lend to each other, only countries lend to each other. Therefore, in the global equilibrium, state-contingent money claims by Home and Foreign have zero excess demand:

$$B(\mathbf{s}) + B^*(\mathbf{s}) = 0. \tag{31}$$

in every state $\mathbf{s}$. The Home resource constraint is given by

$$G[y_h(\mathbf{s}), y_f(\mathbf{s})] = X(\mathbf{s}) + I(\mathbf{s}) + G^d(\mathbf{s}), \tag{32}$$

where $I(\mathbf{s}) = K(\mathbf{s}) - (1-\delta)K(\mathbf{s}_-)$ is domestic capital investment.

The Foreign resource constraint is given by

$$G[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})] = X^*(\mathbf{s}) + I^*(\mathbf{s}) + G^{d*}(\mathbf{s}), \tag{33}$$

where $I^*(\mathbf{s}) = K^*(\mathbf{s}) - (1-\delta)K^*(\mathbf{s}_-)$ is the Foreign country's investment in its own capital stock, and, government spending G^{d*} is given by

$$\begin{aligned} G^{d*}(\mathbf{s}) = {} & [T^*(\mathbf{s}) + (M^*(\mathbf{s}) - M^*(\mathbf{s}_-))\phi^*(\mathbf{s})] \\ & + \tau_X X^*(\mathbf{s}) + \tau_H H^*(\mathbf{s}) + \tau_K(\tilde{r}^*(\mathbf{s}) - \delta)K^*(\mathbf{s}_-). \end{aligned}$$

We also assume that $T^*(\mathbf{s}) = -(M^*(\mathbf{s}) - M^*(\mathbf{s}_-))\phi(\mathbf{s})$.

Market clearing for the intermediate goods must hold:

$$zF[K(\mathbf{s}_-), H(\mathbf{s})] = y_h(\mathbf{s}) + y_h^*(\mathbf{s}) \tag{34}$$

$$z^*F[K^*(\mathbf{s}_-), H^*(\mathbf{s})] = y_f^*(\mathbf{s}) + y_f(\mathbf{s}) \tag{35}$$

Definition 1. *A stationary Markov monetary equilibrium (SME), given any feasible monetary policy rule* (ψ, ψ^*)*, is a set of time-invariant maps consisting of*

E1. *strictly positive pricing functions* (ϕ, ϕ^*, e) *and* (w, r, w^*, r^*, Q)*,*

E2. *transition laws* $(\mathcal{G}, \varphi)$ *and* $(\mathcal{G}^*, \varphi^*)$*,*

E3. *value functions* V, W *and* V^*, W^**,*

E4. *CM decision rules* $(X, X^*, m, m^*, b, k, b^*, k^*)$*, and*

E5. *DM terms of trade (decision rules),* $(d, q, \breve{q})$ *and* $(d^*, q^*, \breve{q}^*)$*,*

such that:

(1) *given prices (E1), the value functions V and W satisfy the functional equations (1), (2), (3), and (19) and symmetrically V^*, W^* solve the Foreign country counterpart problems;*
(2) *given the value functions V and W, and prices (E1), the decision rules E4 solve (1), (2), (3) in the CM, for the Home country and symmetrically for the Foreign country, given V^* and W^*;*
(3) *Firms optimize: (17) and (18);*
(4) *given the value functions W and V, the decision rules E5 solve and (23), (25), and (26) in the DM, and symmetrically for the Foreign country, given W^*;*
(5) *The government budget constraint (21) is satisfied for Home and symmetrically for Foreign.*
(6) *Markets clear in the CM and CM*: (31), (32) and (33), where $m = M$, $b = B$ and $k = K$, and $m^* = M^*$, $b^* = B^*$ and $k^* = K^*$.*

3.4. **Other variable definitions.** Since the model features a DM sector that is akin to a nontraded goods sector, we will define a relevant price index, which will be used toward the construction of a real exchange rate definition. First we define a DM price index as the convex combination of the pricing outcome in monetary and credit trades:

$$p_{DM} := \kappa\tilde{p} + (1-\kappa)\breve{p}.$$

The foreign counterpart will be p^*_{DM}. Denote the aggregate DM consumption as

$$q_{DM} := \kappa q + (1-\kappa)\breve{q}.$$

Now we can define our measure of aggregate price index (or output deflator) as

$$P_Y = \zeta\phi^{-1} + (1-\zeta)p_{DM},$$

where

$$\zeta = \frac{X}{X + \sigma q_{DM}},$$

is the CM consumption share in total domestic consumption. Note that this share is time-varying in the sense that it is dependent on the aggregate state $\mathbf{s}$. The foreign price index is defined analogously as P^*_Y. Now we define the real exchange rate as

$$RER(\mathbf{s}) := \frac{e(\mathbf{s})P^*_Y(\mathbf{s})}{P_Y(\mathbf{s})}. \tag{36}$$

4. Implications for Exchange Rate Dynamics

We now analyze the implication of the assumption of anonymity ($0 < \kappa \leq 1$), for exchange rate dynamics. For ease of notation and exposition, and without loss of generality, we consider

$\kappa = 1$ (*i.e.* extreme anonymity in the DM) for now and $\tau_X = \tau_H = \tau_K = 0$. Using the first-order conditions in the CM and DM, the corresponding envelope conditions, and imposing equilibrium, we can derive a set of stochastic Euler functional equations necessary for characterizing a *stationary Markov monetary equilibrium* (SME). We can write the SME conditions as ones that characterize the solutions as $\mathbf{s}$-dependent processes.[14]

First, from (6), we can easily deduce that in equilibrium, $X(\mathbf{a}, \mathbf{s}) = X(\mathbf{s})$, and, $X^*(\mathbf{a}^*, \mathbf{s}) = X^*(\mathbf{s})$, for all $\mathbf{s}$. Also, $q(m, k, \mathbf{s}) = q(M, K, \mathbf{s}) \equiv q(\mathbf{s})$, and, $q^*(m^*, k^*, \mathbf{s}) = q^*(M^*, K^*, \mathbf{s}) \equiv q^*(\mathbf{s})$. Together with (7) and (27), we have the SME version of the Euler functional equation for optimal money holdings in the Home country:

$$U_X[X(\mathbf{s})] = \beta \mathbb{E}_\lambda \left\{ U_X[X(\mathbf{s}_+)] \frac{\phi(\mathbf{s}_+)}{\phi(\mathbf{s})} \left[(1 - \sigma) + \sigma \frac{z_+ u_q[q(\mathbf{s}_+)]}{c_q[q(\mathbf{s}_+)/z_+, K(\mathbf{s})]} \right] \right\}, \tag{37}$$

where, $\mathbb{E}_\lambda$ denotes the expectation operator with respect to the conditional distribution $\lambda(\mathbf{s}, \cdot)$, and, the term in the square brackets is the expected (with respect to ν_h) one-period nominal gross return on money holding. There is an equivalent condition for the foreign country.

Second, since in equilibrium, $X(\mathbf{a}, \mathbf{s}) = X(\mathbf{s})$ for all $\mathbf{s}$, along with (8) and (28), we then have an Euler equation for optimal Home bond holdings:

$$\begin{aligned} Q(\mathbf{s}_+|\mathbf{s}) &:= \left[\int_{\mathbf{a}_+} Q(\mathbf{a}_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s}) \mu_h(\mathbf{s}_+, d\mathbf{a}_+) \right] \lambda(\mathbf{s}, d\mathbf{s}_+) \\ &= \beta \frac{U_X[X(\mathbf{s}_+)]}{U_X[X(\mathbf{s})]} \frac{\phi(\mathbf{s}_+)}{\phi(\mathbf{s})} \lambda(\mathbf{s}, d\mathbf{s}_+), \qquad \forall \mathbf{s}, \mathbf{s}_+. \end{aligned} \tag{38}$$

Third, Foreign agents would also have a first order condition for bonds similar to (38), which, in Home currency terms is:

$$\begin{aligned} Q(\mathbf{s}_+|\mathbf{s}) &:= \left[\int_{\mathbf{a}^*_+} Q(\mathbf{a}^*_+, \mathbf{s}_+|\mathbf{a}, \mathbf{s}) \mu_f(\mathbf{s}_+, d\mathbf{a}^*_+) \right] \lambda(\mathbf{s}, d\mathbf{s}_+) \\ &= \beta \frac{U_X[X^*(\mathbf{s}_+)]}{U_X[X^*(\mathbf{s})]} \frac{\phi^*(\mathbf{s}_+)}{\phi^*(\mathbf{s})} \frac{e(\mathbf{s})}{e(\mathbf{s}_+)} \lambda(\mathbf{s}, d\mathbf{s}_+), \qquad \forall \mathbf{s}, \mathbf{s}_+. \end{aligned} \tag{39}$$

From (6), (9) and knowing V_K, we have an Euler equation for optimal Home capital holdings:

$$\begin{aligned} &U_X[X(\mathbf{s})] = \\ &\quad \beta \mathbb{E}_\lambda \left\{ U_X[X(\mathbf{s}_+)] \left[(1 + r(\mathbf{s}_+) - \delta) - \sigma \frac{\gamma[q(\mathbf{s}_+), K(\mathbf{s}), z_+]}{U_X[X(\mathbf{s}_+)]} \right] \right\}. \end{aligned} \tag{40}$$

There is also a symmetric characterization for the foreign country.

[14] The full details are given in a separate Appendix available from the authors. Recall that in any equilibrium, agents end up choosing the same asset allocations regardless of their personal state. Thus, with a slight abuse of notation, we drop the dependency on aggregate state variables such as $\mu_i(\mathbf{s}, \cdot)$, $i \in \{h, f\}$, from the definition of $\mathbf{s}$ in equilibrium. In other words, the Euler equations below will have the appearance as though they were—and indeed they are—characterizing equilibrium of some representative agent model.

4.1. **Inspecting the mechanism.** Equating (38) and (39) and iterating, we have

$$\frac{U_X[X(\mathbf{s})]}{U_X[X(\mathbf{s}_0)]}\frac{\phi(\mathbf{s})}{\phi(\mathbf{s}_0)} = \frac{U_X[X^*(\mathbf{s})]}{U_X[X^*(\mathbf{s}_0)]}\frac{e(\mathbf{s}_0)}{e(\mathbf{s})}\frac{\phi^*(\mathbf{s})}{\phi^*(\mathbf{s}_0)}, \tag{41}$$

where $\mathbf{s}_0$ is the initial aggregate state. Assume that the initial condition, given by

$$\kappa_0 := \frac{e(\mathbf{s}_0)U_X[X(\mathbf{s}_0)]\phi(\mathbf{s}_0)}{U_X[X^*(\mathbf{s}_0)]\phi^*(\mathbf{s}_0)}$$

is fixed. We can re-write the expression in (41) as the equilibrium determination of the nominal exchange rate:

$$e(\mathbf{s}) = \kappa_0 \frac{U_X[X^*(\mathbf{s})]}{U_X[X(\mathbf{s})]}\frac{\phi^*(\mathbf{s})}{\phi(\mathbf{s})}. \tag{42}$$

This warrants some remark. Up to this point, in terms of equilibrium complete state-contingent money claims, we have derived a standard complete markets (in terms of the CM) result for the nominal exchange rate [see e.g. Chari, Kehoe, and McGrattan, 2002]. What equation (42) says is that the nominal exchange rate, at each state of the world, is proportional to the within-period the relative value of the marginal rate of substitution of the general good between Home and Foreign consumers.

Note however, in equilibrium, the DM price-taking protocol implies that buyers' marginal utility value of holding domestic currency must equal sellers' marginal utility cost of producing good q, where by anonymity, must be purchased with money:

$$U_X[X(\mathbf{s})]\phi(\mathbf{s})M(\mathbf{s}) = \frac{1}{z}c_q\left(\frac{q(\mathbf{s})}{z}, K(\mathbf{s}_-)\right)q(\mathbf{s}) \equiv g[q(\mathbf{s}), K(\mathbf{s}_-), z]. \tag{43}$$

In terms of stationary variables – i.e. normalizing by $M(\mathbf{s}_-)$ – and assuming logarithmic utility for U, we have:

$$\frac{\hat{\phi}(\mathbf{s})}{X(\mathbf{s})} = \frac{1}{\exp\{\psi_t\}}c_q\left(\frac{q(\mathbf{s})}{z}, K(\mathbf{s}_-)\right)\frac{q(\mathbf{s})}{z} \equiv \frac{1}{\exp\{\psi_t\}}g[q(\mathbf{s}), K(\mathbf{s}_-), z], \tag{44}$$

where $\hat{\phi}(\mathbf{s}) := \phi(\mathbf{s})M(\mathbf{s}_-)$ and $M(\mathbf{s})/M(\mathbf{s}_-) = \exp\{\psi_t\}$.

In contrast now, consider a version of our model where money is introduced via a cash-in-advanced (CIA) constraint, *à la* Cooley and Hansen [1989]. In a monetary equilibrium where the CIA constraint binds almost surely, we would have:

$$\frac{\hat{\phi}(\mathbf{s})}{X(\mathbf{s})} = \frac{1}{\exp\{\psi_t\}}. \tag{45}$$

The interpretation in the CIA version is obviously quite different. In such an economy, agents are constrained to hold money to buy goods by assumption. Equation (45) implies that a positive increase in money supply (on the right) must be followed by a virtually one-for-one increase in the price level (or decrease in the value of a dollar, $\hat{\phi}$), if equilibrium consumption X is smooth (or equivalently if agents are risk-averse and markets are complete). In short, the relative price of a unit of X is extremely flexible in response to a monetary shock. If so, from the nominal exchange rate determination condition in (42), we can immediately deduce that there would be very little volatility in the nominal exchange rate. Hence there would be very little connection between the nominal and the real exchange rates as well, by the definition of the real exchange rate.[15]

Consider now our model with extreme anonymity ($\kappa = 1$). Anonymity implies that the equilibrium condition (44) must hold. With log utility, we have a direct comparison between our model and a model with the CIA constraint (45). In contrast, even in the presence of consumption smoothing, the DM equilibrium pricing condition (44) implies that an increase in money supply need not be followed by a one-for-one increase in the price level, or a decrease in the value of money. Holding the conditional expectations on the right of (37) constant, a positive monetary injection means that current q will increase, on the left side of the equilibrium money Euler equation (37). As current q increases immediately, this has an opposing effect to an increase in money supply. That is, on the one hand, an increase in money supply has a tendency to reduce the marginal utility value of holding a dollar (the left side of (44)), an increase in q tends to increase the utility value of that dollar purchasing the special good q (the right side of (44)). Depending on the nature of the DM pricing protocol and parametrization – *i.e.* the shape of g, it may be that the value of a dollar $\hat{\phi}$ need not fall as much as the increase in money supply. In other words, it may be possible that the equilibrium pricing process will appear rather rigid or unresponsive as an equilibrium outcome, rather than being an assumption.

Consider also a supply-side or technology shock, z. An increase z, has a tendency to raise the current marginal product of labor and hence labor demand in the CM. Equating (6) and (18), we have a condition for equilibrium labor market clearing in the CM. From this, we can see that if consumption increases but by not as much as income, then labor allocation would also increase. This would imply an increase in current CM investment into productive capital stock next period. Since $c(q/z, K)$ is the dual cost function to an homogeneous of degree one production technology in the DM, we can deduce that an increase in z will lower the marginal cost of producing q. This will, in turn, lower the term on the right of the equilibrium monetary pricing condition (44). However, the technology shock also affects the left side of (44) via raising the marginal product of labor, and hence lowering the marginal utility of X, $U_X(X)$. Again, depending on the shape of g, the value of a dollar, $\hat{\phi}$, need not be so responsive to a technology shock. Therefore, consistent with the nominal exchange rate determination condition (42), the nominal exchange rate ought to be quite volatile too. Since the real exchange rate in our two-sector model is defined by (36), we would expect the real exchange rate to co-move with the nominal exchange rate.

[15]This point has previously been verified by the earlier work of Schlagenhauf and Wrase [1995] in the context of a two-country CIA monetary model.

Finally, we may expect that shocks are also propagated as persistent equilibrium relative pricing processes too. There are two reasons for this conjecture, independent of the intrinsic persistence in the shock processes. The first reason arises from equilibrium forward-looking behavior and risk aversion. We assumed that the utility functions U and u, along with the production functions F and $\tilde{F}$ are strictly concave, where $c = \tilde{F}^{-1} \circ K$ is strictly convex. These would imply, from the intertemporal trade-offs in (37) and (40), that the equilibrium allocation for $\{X(\mathbf{s}_t), q(\mathbf{s}_t)\}_{t\in\mathbb{N}}$ would exhibit some intertemporal and across-state smoothing, and hence persistence in its equilibrium time-series properties. Since the equilibrium condition (43) pinning down the value of money $\hat{\phi}$ will depend on X and q, $\hat{\phi}$ may inherit some persistence too. The second reason may arise, as a result of backward-looking dynamics. Specifically, if we assume that capital is used for production in the DM ($\varpi > 1$), then the g function in (43) would admit K_t as an argument in equilibrium. Since $\{K(\mathbf{s}_t)\}_{t\in\mathbb{N}}$ is inherently a backward-looking process, some of its persistence would also matter for the persistence for the relative price $\hat{\phi}$. The persistence in relative prices, along with the nominal exchange rate via equation (42), may imply persistence in the nominal and real exchange rates.

In the next sections, we will validate these equilibrium implications for the exchange rate dynamics.

5. Computational Exercise

For our numerical exercise, we consider the following specific functions to represent the model primitives. In the CM, per-period preferences and technology are represented by

$$U(X) = B\frac{X^{1-\gamma}-1}{1-\gamma}, \qquad zF(K,H) = zK^{\alpha}H^{1-\alpha},$$

respectively, where $B > 0$, $\gamma > 0$, and $\alpha \in (0,1)$. The symmetric description holds for the Foreign country. Note however, the notation for the final goods production function G is such that

$$G(y_h, y_f) = \left[\vartheta(y_h)^{\frac{1}{\epsilon}} + (1-\vartheta)(y_f)^{\frac{1}{\epsilon}}\right]^{\epsilon},$$

for the Home country, and,

$$G(y_f^*, y_h^*) = \left[\vartheta(y_f^*)^{\frac{1}{\epsilon}} + (1-\vartheta)(y_h^*)^{\frac{1}{\epsilon}}\right]^{\epsilon},$$

for the Foreign country, where $\vartheta \in (0,1)$ and $-\infty \leq 1/\epsilon \leq 1$. The elasticity of substitution between the inputs to G is given by $\sigma_\epsilon = \epsilon/(\epsilon-1)$. These functional forms are quite standard in models with international trade in intermediate goods [see e.g. Heathcote and Perri, 2002; Chari, Kehoe, and McGrattan, 2002].

In the DM, per-period preferences and technology are respectively represented by

$$u(q) = C\frac{(q+\underline{q})^{1-\eta} - b^{1-\eta}}{1-\eta}, \qquad c(q,K) = q^{\varpi}(K)^{1-\varpi}$$

TABLE 1. Calibration and Parameterization

Parameter	Values	Remarks
β	0.99	Fixed
$\eta = \gamma$	1	Fixed
δ	0.025	I/K
α	1/3	Total capital income share, 1/3
A	0.4858	Total labor hours fraction, 1/3
ϖ	1.2766	$K/Y = 8.92$ per quarter (2.23 per annum)
σ	0.13	Real money demand interest elasticity, -0.23 (AWW)
B	0.1686	Non-traded good consumption share, 0.50
ϑ	0.9397	Share of imports in net exports (CKM)
ϵ	3	Estimated, CKM, BKK
κ	0.15	Estimated, AWW
τ_K	0.548	Estimated, AWW
τ_H	0.242	Estimated, AWW
τ_X	0.069	Estimated, AWW

Notes:
(a) Aruoba, Waller, and Wright [2009]: (AWW).
(b) Backus, Kehoe, and Kydland [1994]: (BKK).
(c) Chari, Kehoe, and McGrattan [2002]: (CKM).

where $C = 1$, without loss of generality, $\eta > 0$ and $\varpi \geq 1$. We set $\underline{q} = 0$ if DM trade is determined by competitive price taking, and $\underline{q} \searrow 0$ in the case of DM bargaining. The latter assumption is required for a well-defined outside-option value in the bargaining problem [see *e.g.* Lagos and Wright, 2005].

5.1. **Baseline model calibration.** Table 1 summarizes the baseline parameter values for the model. To discipline our numerical exercise, we calibrate the model with a quarterly frequency to match long run stylized facts. First, we discuss parameters that can be easily estimated or fixed indepedently. Similar to Aruoba, Waller, and Wright [2009], we calibrate α to match the target of labor share in output, which is about 0.7 in the data [see also Aruoba, 2010]. We fix $\delta = 0.1$ as estimated in Heathcote and Perri [2002] for a two country model. Following Aruoba, Waller, and Wright [2009] and Aruoba [2010], we calibrate σ to match the long-run money demand semi-elasticity with respect to the nominal interest rate, where money is defined by M1 for the U.S. This elasticity is about -0.23. The risk aversion parameters η and γ imply that both U and u are natural log functions of X and q, respectively. This restriction is required for the baseline model to have a balanced growth path, since the per-period utility function is linearly separable in consumption and leisure [see Waller, 2010]. The constant marginal taxes on capital, labor and CM-consumption, $(\tau_K, \tau_H, \tau_X) = (0.548, 0.242, 0.069)$, are chosen as in Aruoba, Waller, and Wright [2009]. The estimate of ϑ is from Backus, Kehoe, and Kydland [1994].

Second, we calibrate simultaneously the remaining parameters (A, B, ϖ) to match the targets of proportion of total hours worked (DM and CM aggregate), H_{tot}, a measure of non-traded consumption goods share in total consumption, NTS, and the long run capital output ratio, K/Y. The

TABLE 2. Percentage standard deviation relative to output

	Data	PT	(% data)	CKM (% data)*	HP (% data)*
Nominal E.R., e	3.34	4.82	144	[88, 99]	n.a.
Real E.R., RER	3.36	2.34	70	[94, 114]	100
Consumption, C	0.72	0.61	85	[97, 111]	[63, 65]
Investment, I	2.70	1.82	67	[46, 60]	[73, 98]
Hours, H_{tot}	0.83	0.46	55	[53, 70]	[42, 48]

Notes:
(a) Percentage of authors' data statistics accounted for by authors' models.*
(b) Chari, Kehoe, and McGrattan [2002] (CKM).
(c) Heathcote and Perri [2002] (HP) model real business cycles.

value of H_{tot} is roughly 0.33, which is standard. This value can be thought of as pinning down the marginal utility of labour parameter A. B is calibrated, in this model, to match a DM consumption (interpreted as a nontradable good in this model) share of total consumption to be close to 0.50 for the U.S., a share estimated by Stockman and Tesar [1995]. This is in contrast to the closed-economy models in Aruoba, Waller, and Wright [2009] and Aruoba [2010], where intuitively, B is calibrated to match the velocity of money. The target capital-output ratio, K/Y, is 2.23 in annual terms. Given other parameters, this ratio can be thought of as pinning down the calibration for ϖ from the Euler equation characterizing equilibrium capital accumulation along the steady state path. The calibrated value of $\varpi > 1$, implies that the more capital is installed for use in the DM production, the lower the cost of producing a unit of DM output q. By duality, this implies that capital is a complementary input to labor effort in DM production.

In the baseline model, we assume that all the TFP levels (and their shocks), in both CM and DM, are uncorrelated with each other [see also Chari, Kehoe, and McGrattan, 2002]. In parameterizing the exogenous TFP autocorrelation parameters (ρ_Z, ρ_{Z^*}) we follow Chari, Kehoe, and McGrattan [2002]. The money supply growth stochastic processes are the estimates from Schlagenhauf and Wrase [1995].

6. INTERNATIONAL BUSINESS CYCLE FEATURES

In this section, we discuss the business cycle dynamics of the calibrated baseline model. We report the quantitative predictions of our benchmark model (labelled "PT" in the tables) relative to a class of business cycle models with sticky prices considered by Chari, Kehoe, and McGrattan [2002] (labelled CKM in the tables), and a real business cycle model of Heathcote and Perri [2002] (HP in the tables).

Hereinafter, when we refer to aggregate or total consumption (C), output (Y) or labor (H_{tot}) variables, we mean the real allocations of these variables in both the DM and the CM in our model,

TABLE 3. Autocorrelations and cross-correlations

	Data	PT	PT (% data)	CKM (% data)*
Autocorrelation:				
Nominal E.R., e	0.83	0.66	80	[53, 80]
Real E.R., RER	0.84	0.66	79	[70, 80]
Consumption, C	0.87	0.82	94	[54, 68]
Investment, I	0.90	0.78	87	[52, 66]
Hours, H_{tot}	0.94	0.92	98	[53, 76]
Output, Y	0.89	0.79	89	[56, 80]
Contemporaneous correlation:				
(RER, e)	0.99	0.99	100	[75,88]
(RER, NX)	0.14	0.17	121	[534,628]

Notes:
(a) Percentage of authors' data statistics accounted for by authors' models.*
(b) A negative sign indicates a counterfactual direction in the model-data accounting.*
(c) Chari, Kehoe, and McGrattan [2002] (CKM) consider several model variations.
(d) Heathcote and Perri [2002] (HP) did not report these statistics.

where the implicit deflator is the output deflator P_Y, as constructed previously in Section 3.4. Aggregate investment (I) and net exports (NX) will be real variables in terms of aggregate goods with price index P_Y.

As we can see from Table 2, the benchmark model can account for the volatilities of the key business cycle data for the U.S. quite well.[16] In particular, the model can account for up to 85% of aggregate consumption volatility, 67% of the volatility in domestic investment, and about 55% of total labor volatility. The model over-predicts the nominal exchange rate volatility by 44% but accounts for a substantial amount of the real exchange rate volatility (70%). Consider the last two columns in Table 2. Relative to previous accounts by Chari, Kehoe, and McGrattan [2002] (various versions of sticky price and/or wages model) and Heathcote and Perri [2002] (real business cycle model with exogenous financial autarky), our more does quite well.

Overall, in terms of the nominal and real exchange rate volatilities, the model is able to reproduce qualitatively the observation that both exchange rates are much more volatile than U.S. GDP. As opposed to Chari, Kehoe, and McGrattan [2002] and Heathcote and Perri [2002], our benchmark model does not rely on large relative risk aversion parameters, sticky prices nor imperfections in international risk sharing to generate volatility.[17] Furthermore, in contrast, standard flexible price two-country CIA models [see Schlagenhauf and Wrase, 1995] are unable to reproduce any realistic volatilities in the real and nominal exchange rates.

[16]Appendix A contains the description of our data.

[17]On the other hand, the competitive equilibrium in our model features incomplete markets as a result of idiosyncratic shocks to agent types each period as they enter the DM. Since there is a link between the DM and CM outcomes via capital, not all consumption risk can be fully insured.

Next, consider the first order autocorrelation coefficients of the equilibrium processes in Table 3. In terms of consumption, investment, labor allocation, and output, the model matches the empirical persistence in the data quite well, and much better than Chari, Kehoe, and McGrattan [2002]. However, in terms of the real and nominal exchange rates, the model under-accounts for the persistence observed in the data by about 20%. However, the baseline model is able to do just as well as the models of Chari, Kehoe, and McGrattan [2002], without requiring any exogenous sticky-price assumption.

In terms of the other open-economy correlations in the data, the model is able to account for the mild positive correlation between the real exchange rate and net exports in the data. Moreover, the model is able to generate a real-nominal exchange rate correlation that is very close to the data. To see why, we consider the partial explanations given in Figures 2 and 3. Figure 2 depicts the impulse response of the components of the real-exchange-rate definition in the model, $RER := eP_Y/P_Y^*$ to a 1% total factor productivity shock in the home country. Figure 3 considers that of a 1% home money supply growth shock. The resulting dynamics of the relative cross-country aggregate price deflators are such that they are not so sensitive to technology shocks. By definition then, the dynamics of the real exchange rate must be tracking that of the nominal exchange very well. Hence the near perfect correlation between the two. In standard sticky-price models [see e.g. Chari, Kehoe, and McGrattan, 2002], the assumption of price stickiness plays a similar, but more obvious, role. However, in our model, this appears to be an equilibrium outcome arising from the DM anonymity assumption and its resulting restriction of asset and relative pricing dynamics. These figures thus confirm our conjecture in Section 4.

6.1. **Inspecting the Mechanism: Baseline with DM Price-taking.** Recall that in Section 4, we provided the explanation of the potential effects of the assumptions of anonymity (and its resulting monetary equilibrium determination) and capital complementarity on relative pricing processes, and therefore equilibrium exchange rates. In this section, we revisit our explanations, by conducting some experiments to identify the role of each of these mechanisms.

Table 4 summarizes these experiments, which are: (i) Benchmark ($\kappa > 0$, $\varpi > 1$): the baseline monetary equilibrium with DM price-taking assumption; (ii) Limit ($\kappa = 0$): No anonymity (or equivalently a two-sector traded/non-traded goods real business cycle equilibrium); (iii) Benchmark variation ($\kappa > 0$, $\varpi = 1$): case (i) without DM capital service; and (iv) Limit ($\kappa = 0$, $\varpi = 1$): No-anonymity version of (iii).

Consider the limit economy (ii) with pure credit trades ($\kappa = 0$) in the DM. This case shuts down completely the role of anonymity and hence monetary friction. This limit economy also identifies a remainder structure: a (separable-utility) version of standard two-sector real-business-cycle model with traded and nontraded goods. However, as column (ii) versus column (i) in Table 4 show, the

TABLE 4. Inspecting the mechanism: Frictions

	DM capital complementarity, $\varpi > 1$		No DM capital complementarity, $\varpi = 1$	
	(i) Anonymity* ($\kappa > 0$)	(ii) No Anonymity ($\kappa = 0$)	(iii) Anonymity ($\kappa > 0$)	(iv) No Anonymity ($\kappa = 0$)
Standard deviation:				
Nominal E.R., e	4.82	n.a.	0.56	n.a.
Real E.R., RER	3.36	0.85	0.18	0.93
Consumption, C	0.61	0.70	0.01	0.74
Investment, I	1.82	2.34	0.04	2.78
Hours, H_{tot}	0.46	0.64	0.05	0.57
Autocorrelation:				
Nominal E.R., e	0.66	0.65	0.65	n.a.
Real E.R., RER	0.66	0.67	0.67	0.99
Consumption, C	0.82	0.78	0.78	0.99
Investment, I	0.78	0.30	0.30	0.91
Hours, H_{tot}	0.92	0.87	0.87	0.89
Output, Y	0.79	0.60	0.60	0.95

* Benchmark calibrated model.

limit real traded/non-traded goods model alone cannot account for the RER stylized fact: That the RER is more volatile than U.S. output.

Note that columns (ii) and (i) of Table 4 represent economies with capital linking both the DM (nontraded good sector) and the CM (traded good sector). We would also like to see what additional contribution the assumption of capital utilization in the DM (nontraded good sector) plays in generating the excess-volatility stylized fact of the RER in the models. This exercise is shown in Columns (iii) and (iv) of Table 4.

Again, the same pattern arises, without monetary frictions, and hence a monetary equilibrium, the limit non-monetary economy in Column (i) of Table 4 cannot account for the stylized fact of excess-volatility in the real exchange rate. In contrast, however, the monetary frictions result in lower persistence of the RER relative to the limit non-monetary economies.

Thus we have verified that, relative to our baseline calibrated model, the key informational friction of anonymity is not only a means of introducing money into models after Lagos and Wright [2005], but they also matter for stochastic equilibrium relative pricing dynamics. In our case of the DM price-taking protocol, our g function indeed is able to produce what we conjectured from analyzing the model's SME conditions in Section 4.

6.2. **Alternative DM Nash bargaining model.** For completeness, we also consider Nash bargaining, originally used in Lagos and Wright [2005], as an alternative DM pricing mechanism. The interpretation now is that agents are bilaterally matched in a random fashion with $\sigma\kappa$ being the joint probability of the event that an agent meets another agent who is able to produce the special good she wants, and, that trade is anonymous. With identical probability $\sigma\kappa$ an agent meets another who wishes to buy the special good she can produce. Alternatively, similar events (agent as

TABLE 5. Data and alternative equilibrium statistics

	Data	DM Price Taking	DM Nash Bargaining
Standard deviation:			
Nominal E.R., e	3.34	4.82	1.40
Real E.R., RER	3.36	2.34	1.88
Consumption, C	0.72	0.61	1.20
Investment, I	2.70	1.82	1.19
Hours, H_{tot}	0.83	0.46	0.15
Autocorrelation:			
Nominal E.R., e	0.83	0.66	0.67
Real E.R., RER	0.84	0.66	0.73
Consumption, C	0.87	0.82	0.70
Investment, I	0.90	0.78	0.84
Hours, H_{tot}	0.94	0.92	0.93
Output, Y	0.89	0.79	0.72
Contemporaneous correlation:			
(RER, e)	0.99	0.99	-0.98
(RER, NX)	0.14	0.17	0.02

buyer or as seller) which are monitored, each occur with probability $\sigma(1-\kappa)$. Thus with probability $1-2\sigma$ an agent leaves the DM with no exchange.[18]

We calibrate this alternative model to the same empirical targets as in the benchmark model. However, we now have an additional parameter θ representing the common bargaining strength of the buyer in both monetary and credit exchanges. Following Aruoba, Waller, and Wright [2009], we calibrate this parameter, jointly with the others, to match a steady state aggregate pricing markup of around 33%.

The business cycle dynamics of this alternative model are reported in Table 5. Qualitatively, this version of the model is able to account for the observed excess volatility and persistence in the nominal and real exchange rates. However, these come at a cost of a counterfactually volatile consumption and investment process (in excess of output volatility). Also, the real and nominal exchange rates are negatively correlated.

7. CONCLUSION

In this paper, we examine whether a flexible price, two-country, search theoretic model of money is able to account for the empirical regularities observed in U.S. real and nominal exchange rate dynamics. We propose a two-country version of Aruoba, Waller, and Wright [2009] where international trade and asset flows occur in the model's Walrasian centralized markets.

[18]The characterization of a monetary equilibrium under Nash bargaining is quite standard [see e.g. Aruoba, Waller, and Wright, 2009; Aruoba, 2010] and can be found in a separate appendix to this paper.

There are two key mechanisms at work in this model that help amplify and propagate international business cycle shocks. The first mechanism is anonymity. This friction induces asset market incompleteness in the sense that individuals are unable to fully insure against their stochastic trading opportunities in the decentralized markets (DM). The second mechanism is the notion of capital complementarity. The latter mechanism provides for an additional return on capital which places additional restriction on the equilibrium asset pricing relations with respect to money and capital.

We show that the relative pricing dynamics of the baseline model behave in such a way that cross-country aggregate relative prices are non-volatile and persistent. This contributes to the excess volatility and persistence in the real and nominal exchange rate. Without requiring exogenous price-stickiness, we are also able to rationalize near perfect positive correlation between the real and nominal exchange rate. Thus monetary friction, in the sense of Lagos and Wright [2005], is more than just a vehicle for a theoretical foundation of money. In a stochastic two-country environment, it restricts asset pricing relations such that the model is able to account for the stylized facts on real and nominal exchange rate fluctuations.

Appendix A. Data

We focus on quarterly data spanning from Quarter 1 of 1975 to Quarter 4 of 2004. Following Heathcote and Perri [2002] we measure employment H_{tot} using the OECD MEI Civilian Employment Index. We obtain measures of the U.S. nominal and real effective exchange rates, as proxies for e and RER, respectively, from the International Monetary Fund's International Financial Statistics (IFS). We measure aggregate private consumption (C), investment (I) and net exports (NX) from the OECD Outlook Quarterly database. Real output is just a sum of these components.

References

ALESSANDRIA, G. (2009): "Consumer Search, Price Dispersion, And International Relative Price Fluctuations," *International Economic Review*, 50(3), 803–829.

ALVAREZ, F., AND M. VERACIERTO (2000): "Labor-Market Policies in an Equilibrium Search Model," in *NBER Macroeconomics Annual 1999, Volume 14*, NBER Chapters, pp. 265–316. National Bureau of Economic Research.

ARUOBA, S. B. (2010): "Money, Search and Business Cycles," Forthcoming *International Economic Review*, University of Maryland.

ARUOBA, S. B., C. J. WALLER, AND R. WRIGHT (2009): "Money and Capital: A Quantitative Analysis," Working Papers 2009-031, Federal Reserve Bank of St. Louis.

BACKUS, D. K., P. J. KEHOE, AND F. E. KYDLAND (1994): "Dynamics of the Trade Balance and the Terms of Trade: The J-Curve?," *American Economic Review*, 84(1), 84–103.

BURDETT, K., AND K. L. JUDD (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–69.

CHARI, V. V., P. J. KEHOE, AND E. R. MCGRATTAN (2002): "Can Sticky Price Models Generate Volatile and Persistent Real Exchange Rates?," *Review of Economic Studies*, 69(3), 533–63.

COOLEY, T. F., AND G. D. HANSEN (1989): "The Inflation Tax in a Real Business Cycle Model," *American Economic Review*, 79(4), 733–48.

DORNBUSCH, R. (1976): "Expectations and Exchange Rate Dynamics," *Journal of Political Economy*, 84(6), 1161–1176.

DUFFIE, D., J. GEANAKOPLOS, A. MAS-COLELL, AND A. MCLENNAN (1994): "Stationary Markov Equilibria," *Econometrica*, 62(4), 745–81.

HEAD, A., AND S. SHI (2003): "A Fundamental Theory of Exchange Rates and Direct Currency Trades," *Journal of Monetary Economics*, 50(7), 1555–1591.

HEATHCOTE, J., AND F. PERRI (2002): "Financial Autarky and International Business Cycles," *Journal of Monetary Economics*, 49(3), 601–627.

LAGOS, R., AND R. WRIGHT (2004): "A Unified Framework for Monetary Theory and Policy Analysis," Staff Report 346, Federal Reserve Bank of Minneapolis.

——— (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3), 463–484.

LESTER, B., A. POSTLEWAITE, AND R. WRIGHT (2008): "Information, Liquidity and Asset Prices," PIER Working Paper Archive 08-039, Penn Institute for Economic Research, Department of Economics, University of Pennsylvania.

LI, Y., AND G. ROCHETEAU (2009): "Liquidity Constraints," Discussion paper, U.C. Irvine.

LUCAS, R. J., AND E. C. PRESCOTT (1974): "Equilibrium Search and Unemployment," *Journal of Economic Theory*, 7(2), 188–209.

ROCHETEAU, G., AND R. WRIGHT (2005): "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 73(1), 175–202.

SCHLAGENHAUF, D. E., AND J. WRASE (1995): "Exchange Rate Dynamics and International Effects of Monetary Shocks in Monetary Equilibrium Models," *Journal of International Money and Finance*, 14, 155–177.

STEINSSON, J. (2008): "The Dynamic Behavior of the Real Exchange Rate in Sticky Price Models," *American Economic Review*, 98(1), 519–33.

STOCKMAN, A. C., AND L. L. TESAR (1995): "Tastes and Technology in a Two-Country Model of the Business Cycle: Explaining International Comovements," *American Economic Review*, 85(1), 168–85.

WALLER, C. J. (2010): "Random Matching and Money in the Neoclassical Growth Model: Some Analytical Results," *Macroeconomic Dynamics*, forthcoming.

FIGURE 2. DM Price taking. Real and nominal exchange rates versus relative aggregate prices: 1% Home TFP increase z.

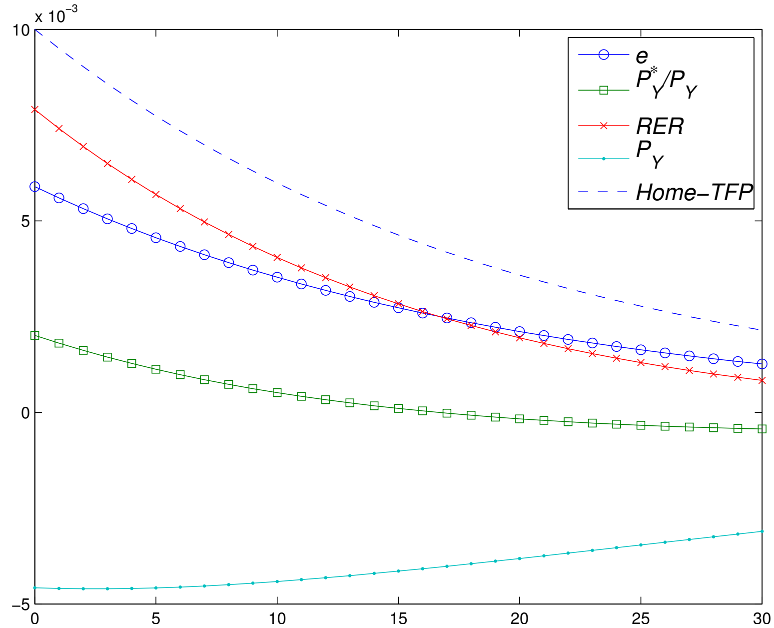
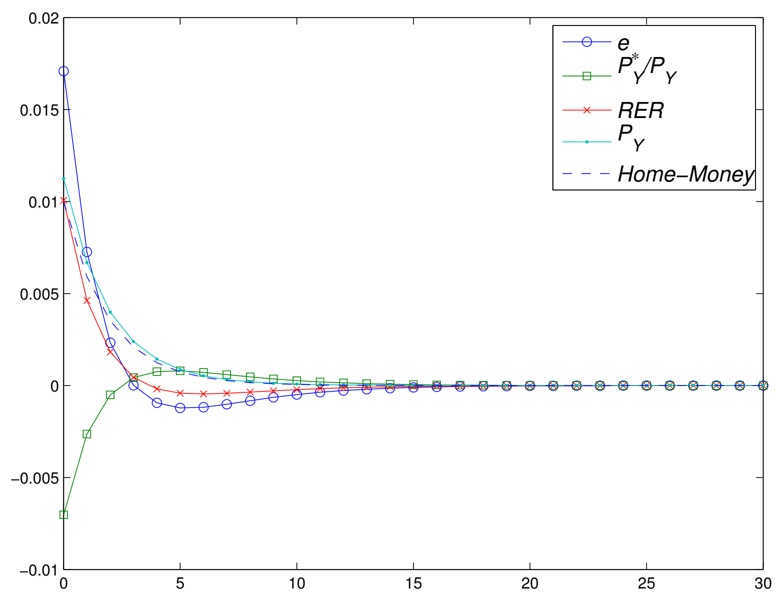


FIGURE 3. DM Price taking. Real and nominal exchange rates versus relative aggregate prices: 1% Home money supply growth increase, ψ.



Supplementary Appendices

Not for publication

APPENDIX B. SME CHARACTERIZATION

Consider a simplification of the model with $\kappa = 1$ and $\tau_K = \tau_X = \tau_H = 0$. Since the processes (ψ) and (ψ^*) are bounded below by zero, this implies that nominal variables, namely M, M^*, ϕ and ϕ^* will grow unboundedly. We can perform a change of variables in the equilibrium conditions for nominal variables as follows. We normalize Home and Foreign nominal variables by $M(\mathbf{s}_-)$ and $M^*(\mathbf{s}_-)$, respectively, such that

$$\hat{\iota}(\mathbf{s}) := \frac{\iota(\mathbf{s})}{M(\mathbf{s}_-)}, \qquad \hat{\iota}^*(\mathbf{s}) := \frac{\iota^*(\mathbf{s})}{M^*(\mathbf{s}_-)}, \qquad \hat{\phi}(\mathbf{s}) := \phi(\mathbf{s})M(\mathbf{s}_-),$$
$$\hat{\phi}^*(\mathbf{s}) := \phi^*(\mathbf{s})M^*(\mathbf{s}_-), \qquad \hat{e}(\mathbf{s}) := \frac{e(\mathbf{s})M^*(\mathbf{s}_-)}{M(\mathbf{s}_-)},$$
$$\hat{P}_h(\mathbf{s}) = P_h(\mathbf{s})/M(\mathbf{s}_-), \qquad \hat{P}_f(\mathbf{s}) = P_f(\mathbf{s})/M(\mathbf{s}_-).$$

Then our SME conditions can be equivalently written as follows. Labor market clearing in the CM in Home and Foreign, respectively, are

$$U_X[X(\mathbf{s})] = \frac{A}{\hat{\phi}(\mathbf{s})\hat{P}_h(\mathbf{s})zF_H\left[K(\mathbf{s}_-), H(\mathbf{s})\right]} \tag{46}$$

$$U_X[X^*(\mathbf{s})] = \frac{A}{\hat{\phi}^*(\mathbf{s})\hat{P}_f^*(\mathbf{s})z^*F_H\left[K^*(\mathbf{s}_-), H^*(\mathbf{s})\right]} \tag{47}$$

The Home resource constraint in equilibrium is given by

$$G(y_h(\mathbf{s}), y_f(\mathbf{s})) = X(\mathbf{s}) + K(\mathbf{s}) - (1-\delta)K(\mathbf{s}_-). \tag{48}$$

The Foreign resource constraint is given by

$$G(y_f^*(\mathbf{s}), y_h^*(\mathbf{s})) = X^*(\mathbf{s}) + K^*(\mathbf{s}) - (1-\delta)K^*(\mathbf{s}_-). \tag{49}$$

Complete international risk sharing entails

$$\frac{\hat{e}(\mathbf{s})\hat{\phi}(\mathbf{s})}{\hat{\phi}^*(\mathbf{s})} = \kappa_0 \frac{U_X[X^*(\mathbf{s})]}{U_X[X(\mathbf{s})]}. \tag{50}$$

where $\kappa_0 = 1$, implying a symmetric initial steady state, without loss of generality.

Aggregate general-good price levels in Home and Foreign, respectively, are pinned down by

$$\frac{A\hat{\phi}(\mathbf{s})}{w(\mathbf{s})}\exp\{\psi\} = g[q(\mathbf{s}), K(\mathbf{s}_-), z], \tag{51}$$

and

$$\frac{A\hat{\phi}^*(\mathbf{s})}{w^*(\mathbf{s})}\exp\{\psi^*\} = g[q^*(\mathbf{s}), K^*(\mathbf{s}_-), z^*]. \tag{52}$$

The equilibrium Euler equations for Home are:

$$g[q(\mathbf{s}), K(\mathbf{s}_-), z] = \beta \mathbb{E}_\lambda \left\{ g[q(\mathbf{s}_+), K(\mathbf{s}), z_+] \exp\{-\psi\} \left[(1-\sigma) + \sigma \frac{u_q[q(\mathbf{s}_+)]}{g_q[q(\mathbf{s}_+), K(\mathbf{s}), z_+]} \right] \right\}, \tag{53}$$

$$U_X[X(\mathbf{s})] = \beta \mathbb{E}_\lambda \left\{ U_X[X(\mathbf{s}_+)] \left[(1 + r(\mathbf{s}_+) - \delta) - \sigma \frac{\gamma[q(\mathbf{s}_+), K(\mathbf{s}), z_+]}{U_X[X(\mathbf{s}_+)]} \right] \right\}. \tag{54}$$

These functional equations determine the equilibrium processes for K and q. Similarly, the equilibrium Euler equations for Foreign are:

$$g[q^*(\mathbf{s}), K^*(\mathbf{s}_-), z^*] = \beta \mathbb{E}_\lambda \left\{ g[q^*(\mathbf{s}_+), K^*(\mathbf{s}), z^*_+] \exp\{-\psi^*\} \times \left[(1-\sigma) + \sigma \frac{u_q[q^*(\mathbf{s}_+)]}{g_q[q^*(\mathbf{s}_+), K^*(\mathbf{s}), z^*_+]} \right] \right\}, \tag{55}$$

$$U_X[X^*(\mathbf{s})] = \beta \mathbb{E}_\lambda \left\{ U_X[X^*(\mathbf{s}_+)] \left[(1 + r^*(\mathbf{s}_+) - \delta) - \sigma \frac{\gamma[q^*(\mathbf{s}_+), K^*(\mathbf{s}), z^*_+]}{U_X[X^*(\mathbf{s}_+)]} \right] \right\}. \tag{56}$$

Note that capital and labor rental pricing functions are given by:

$$r(\mathbf{s}) = \hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) \cdot z F_k[K(\mathbf{s}_-), H(\mathbf{s})], \tag{57}$$

and

$$w(\mathbf{s}) = \hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) \cdot z F_H[K(\mathbf{s}_-), H(\mathbf{s})], \tag{58}$$

for Home, and

$$r^*(\mathbf{s}) = \frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_f(\mathbf{s})}{e(\mathbf{s})} \cdot z^* F_k[K^*(\mathbf{s}_-), H^*(\mathbf{s})], \tag{59}$$

and

$$w^*(\mathbf{s}) = \frac{\hat{\phi}^*(\mathbf{s}) \hat{P}_f(\mathbf{s})}{e(\mathbf{s})} \cdot z^* F_H[K^*(\mathbf{s}_-), H^*(\mathbf{s})], \tag{60}$$

for Foreign, where we have made use of the law of one price for intermediate goods.

Intermediate goods trade and market clearing are given by:

$$\hat{\phi}(\mathbf{s}) \hat{P}_h(\mathbf{s}) = G_{y_h}[y_h(\mathbf{s}), y_f(\mathbf{s})], \tag{61}$$

and

$$\hat{\phi}(\mathbf{s}) \hat{P}_f(\mathbf{s}) = G_{y_f}[y_h(\mathbf{s}), y_f(\mathbf{s})]. \tag{62}$$

for Home, and

$$\frac{\hat{\phi}^*(\mathbf{s})\hat{P}_f(\mathbf{s})}{e(\mathbf{s})} = G_{y_f^*}[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})], \tag{63}$$

and

$$\frac{\hat{\phi}^*(\mathbf{s})\hat{P}_h(\mathbf{s})}{e(\mathbf{s})} = G_{y_h^*}[y_f^*(\mathbf{s}), y_h^*(\mathbf{s})]. \tag{64}$$

for Foreign, where we have again made use of the law of one price for intermediate goods.

Market clearing for intermediate goods are:

$$zF[K(\mathbf{s}_-), H(\mathbf{s})] = y_h(\mathbf{s}) + y_h^*(\mathbf{s}), \tag{65}$$

$$z^*F[K^*(\mathbf{s}_-), H^*(\mathbf{s})] = y_f^*(\mathbf{s}) + y_f(\mathbf{s}). \tag{66}$$

Definition 2. *A stationary Markov monetary equilibrium (with decentralized bargaining) is given by time-invariant functions of* $\mathbf{s}$, *i.e.*

(1) *Consumption functions* $(X, X^*, H, H^*, q, q^*, y_h, y_f, y_f^*, y_h^*)$,
(2) *Savings functions* (K, K^*), *and,*
(3) *Pricing functions* $(w, w^*, r, r^*, \hat{e}, \hat{\phi}, \hat{\phi}^*, \hat{P}_h, \hat{P}_y)$,

that induce bounded stochastic processes satisfying the recursions (46)-(66), given policies $(\psi(\mathbf{s}), \psi^*(\mathbf{s}))$.

Appendix C. Non-monetary Limit Economy

In this appendix, we outline the solution for the limit economy when $\kappa = 0$ in the baseline model. Hence the allocation will be equivalent to a version of a real business cycle model with a traded (CM) and a non-traded (DM) goods sector. Variables are defined in Table 6.

Table 6. Variable definition

Mnemonic	Description
K	Home capital stock
K^*	Foreign capital stock
z	Home TFP state
z^*	Foreign TFP state
μ	Home money supply growth
μ^*	Foreign money supply growth
X	Home CM good
X^*	Foreign CM good
q	Home DM good
q^*	Foreign DM good
a	Home produced intermediate good, Home use
a^*	Home produced intermediate good, Foreign use
b	Foreign produced intermediate good, Home use
b^*	Foreign produced intermediate good, Foreign use

Variety $a + a^*$ (used by Home and Foreign) is produced by the Home country's technology F, and vice-versa for variety $b + b^*$. $c : \mathbb{R}^2_+ \to \mathbb{R}_+$ is the cost function describing the technology of producing q. Let G be the technology that aggregates the inputs (a, b) for the Home country and (b^*, a^*) for the Foreign country into a final general good of the same characteristic as X and X^* respectively.

C.1. **The planner's problem.** We characterize the equilibrium as an equivalent planner's solution. Define an allocation function by

$$\alpha := (X, X^*, H, H^*, q, q^*, K, K^*, a, a^*, b, b^*).$$

Denote s_t as the vector of relevant state variables. Here, we have $s_t := (K, K^*, z, z^*)$. Let $s_t \mapsto J(s_t)$ be the planner's value function. A Pareto allocation $\{\alpha(s_t)\}_{t\in\mathbb{N}}$ in this economy is generated by an α satisfying the following Bellman equation:

$$J(s) = \max_{\alpha} \Big\{ U(X) - AH + U(X^*) - AH^* \\ + \sigma[u(q) - c(q/z, K)] + \sigma[u(q^*) - c(q^*/z^*, K^*)] + \beta\mathbb{E}[J(s')|s] \Big\} \tag{67}$$

subject to

$$K' = G(a, b) + (1 - \delta)K - X, \tag{68}$$

$$K^{*\prime} = G(b^*, a^*) + (1 - \delta)K^* - X^*, \tag{69}$$

$$a + a^* = zF(K, H), \tag{70}$$

$$b + b^* = z^*F(K^*, H^*). \tag{71}$$

Let $(\zeta, \zeta^*, \phi, \phi^*)$ be the state-by-state Lagrange multipliers on the respective constraints above. The first-order conditions for the RHS problem in the Bellman equation are:

$$\begin{aligned}
X &: \quad U_X(X) = \zeta, & X^* &: \quad U_{X^*}(X^*) = \zeta^*, \\
K' &: \quad -\zeta + \beta\mathbb{E}[J_{K'}(s')|s] = 0, & K^{*\prime} &: \quad -\zeta^* + \beta\mathbb{E}[J_{K^{*\prime}}(s')|s] = 0, \\
H &: \quad -A + \phi z F_H(K, H) = 0, & H &: \quad -A + \phi^* z^* F_{H^*}(K^*, H^*) = 0, \\
q &: \quad \sigma[u_q(q) - c_q(q/z, K)/z] = 0, & q^* &: \quad \sigma[u_{q^*}(q^*) - c_{q^*}(q^*/z^*, K^*)/z^*] = 0, \\
a &: \quad \zeta G_a(a, b) - \phi = 0, & a^* &: \quad \zeta^* G_{a^*}(b^*, a^*) - \phi = 0, \\
b &: \quad \zeta G_b(a, b) - \phi^* = 0, & b^* &: \quad \zeta^* G_{b^*}(b^*, a^*) - \phi^* = 0,
\end{aligned}$$

and feasibility conditions are given in (68)-(71).

Under regularity assumptions J is continuously differentiable.[19] Then the envelope conditions, with respect to K and K^*, at an interior maximum are

$$J_K(s) = -\sigma c_K(q/z, K) + \zeta(1-\delta) + \phi[zF_K(K,H)],$$

and

$$J_{K^*}(s) = -\sigma c_{K^*}(q^*/z^*, K^*) + \zeta^*(1-\delta) + \phi^*[z^* F_{K^*}(K^*, H^*)].$$

From these optimality conditions, we have the characterization of a Pareto allocation $\{\alpha(s_t)\}_{t\in\mathbb{N}}$. More precisely, after some straightforward substitution, we have the following definition.

Definition 3. *A **Pareto allocation** $\{\alpha(s_t)\}_{t\in\mathbb{N}}$ is given by a list of allocation functions*

$$\alpha := (X, X^*, H, H^*, q, q^*, K, K^*, a, a^*, b, b^*)$$

satisfying the following conditions:

$$\zeta = U_X(X)$$

$$\zeta^* = U_{X^*}(X^*)$$

$$\zeta = \beta\mathbb{E}\left\{\zeta' \left[G_a(a', b') z' F_K(K', H') + 1 - \delta\right] - \sigma c_K(q'/z', K') \middle| s\right\}$$

$$\zeta^* = \beta\mathbb{E}\left\{\zeta^{*\prime} \left[G_{b^*}(b^*, a^*) z^{*\prime} F_{K^*}(K^{*\prime}, H^{*\prime}) + 1 - \delta\right] - \sigma c_{K^*}(q^{*\prime}/z^{*\prime}, K^{*\prime}) \middle| s\right\}$$

$$A = zF_H(K,H)\zeta G_a(a,b)$$

$$A = z^* F_{H^*}(K^*, H^*)\zeta^* G_{b^*}(b^*, a^*)$$

$$u_q(q) = c_q(q/z, K)/z$$

$$u_{q^*}(q^*) = c_{q^*}(q^*/z^*, K^*)/z^*$$

$$G_a(a,b)\zeta = G_{a^*}(b^*, a^*)\zeta^*$$

$$G_b(a,b)\zeta = G_{b^*}(b^*, a^*)\zeta^*$$

$$zF(K,H) = a + a^*$$

$$z^* F(K^*, H^*) = b + b^*$$

$$G(a,b) = X + K' - (1-\delta)K$$

$$G(b^*, a^*) = X^* + K^{*\prime} - (1-\delta)K^*.$$

[19]Given (i) the state space is a convex and compact Borel subset of $\mathbb{R}^4$; (ii) and appropriate assumptions of the stochastic processes on (z, z^*) – i.e. the transition probability functions have the Feller property; (iii) continuous differentiability of the per-period payoff on the state space; and (iv) given assumptions that F and G are continuous, and define convex production sets, then $J(\cdot, z, z^*)$ is continuously differentiable in (K, K^*) at some (K_0, K_0^*) in the interior of the state space.

Remark 1. *The planner allocates q and q^* efficiently. That is for all states s_t and dates $t \in \mathbb{N}$, the marginal utility of a buyer consuming q in the Home country is equal to a seller's marginal cost of producing it, $u_q(q) = c_q(q/z, K)/z$. Likewise for q^*. This coincides with the outcome of a barter economy if there were no double coincidence of wants problem [see also Lagos and Wright, 2005; Aruoba, Waller, and Wright, 2009].*

Remark 2. *The terms of trade and international relative price for tradable intermediate goods is given by:*

$$\frac{\phi^*}{\phi} = \frac{G_b(a,b)}{G_a(a,b)} = \frac{G_{b^*}(b^*,a^*)}{G_{a^*}(b^*,a^*)}.$$

Remark 3. *Let X^* be the numeraire good. Denote the non-traded special good q share of total consumption as:*

$$\chi := \frac{\sigma q}{\sigma q + X},$$

for the Home country, and

$$\chi^* := \frac{\sigma q^*}{\sigma q^* + X^*},$$

for Foreign. Denote $p_X := U_X(X^*)/U_X(X)$ as the general good real terms of trade. Note that since X^* is numeraire, then $p_X^* := 1$. The relative prices between special and general goods are then

$$p_q := U_X(X^*)/u_q(q),$$

and

$$p_q^* := U_X(X^*)/u_q(q^*),$$

respectively, for the Home and Foreign, special goods. Then the real exchange rate is defined as

$$RER := \frac{\chi^* p_q^* + (1-\chi^*)\cdot 1}{\chi p_q + (1-\chi)p_X}. \tag{72}$$

APPENDIX D. BARGAINING

Our modeling strategy proceeds from the baseline model with decentralized market (DM) price taking, to two alternative bargaining protocols (which have increasing sources of frictions) for determining the terms of trade in the DM. The former baseline environment has the minimal number of frictions introduced into the DM trading environment (i.e. degree of anonymity, κ and the search-matching friction σ). The generalized Nash bargaining (GNB) setup introduces both money (when inflation in some states of nature is away from the Friedman rule) and capital holdup frictions, whenever $0 < \theta < 1$.[20] In this appendix, we outline these two alternatives to the baseline model.

In section D.1 we consider the generalized Nash bargaining solution used originally by Lagos and Wright [2005]. Finally in section E, we detail the nonstochastic steady state conditions in the baseline model, and also how we calibrate a subset of the baseline model's parameters that are not estimated elsewhere. In this section we also show where departures and additions occurs in the case of the GNB alternative model.

D.1. **Generalized Nash bargaining.** In each single-coincidence meeting that occurs with probability $\sigma\kappa$, the money exchanged d and quantity traded q, solve a generalized Nash bargaining problem:

$$\max_{q \in \mathbb{R}_+, d \in [0,m]} \Big\{ [u(q) + W(m_b - d, \cdot) - T_b]^{\theta} \times [-c(q/z, k_s) + W(m_s + d, \cdot) - T_s]^{1-\theta} \Big\}, \tag{73}$$

where $T_b = W(m_b, \cdot)$ and $T_s = W(m_s, \cdot)$ are the respective threat points of the buyer and the seller – i.e. their individual values of entering the next CM with empty trades from the DM. The parameter $\theta \in [0, 1]$ is the bargaining strength of the buyer, and, is also the probability that the buyer gets to make an offer in the subsequent round of an equivalent sequential bargaining game.

By the linearity of the value function W, at each given $\mathbf{s}$, the problem can be further simplified to

$$\max_{q \in \mathbb{R}_+, d \in [0,m]} \left\{ \left[u(q) - \frac{A\phi}{w} d \right]^{\theta} \left[-c(q/z, k_s) + \frac{A\phi}{w} d \right]^{1-\theta} \right\}. \tag{74}$$

D.1.1. *DM monetary exchange.* Consider bilateral single-coincidence trades where money is essential as a medium of exchange. In equilibrium, the constraint $d \leq m_b = m$ binds. So then, a solution to the programming problem in (74) is necessarily and sufficiently given by the decision rules $q(m, k_s, \hat{\mathbf{s}})$ and $d(m, k_s, \hat{\mathbf{s}})$ satisfying:

$$d(m, k_s, \hat{\mathbf{s}}) = m, \tag{75}$$

$$\frac{A\phi}{w} m = \frac{\theta c(q/z, k_s) u_q(q) + (1-\theta) u(q) c_q(q/z, k_s)/z}{\theta u_q(q) + (1-\theta) c_q(q/z, k_s)/z} \equiv g(q, k_s, \hat{\mathbf{s}}). \tag{76}$$

[20]As discussed in Aruoba, Waller, and Wright [2009], if we set $\theta = 1$, the buyer takes all the surplus in a GNB outcome, and this resolves the money holdup inefficiency on the buyer's part, but creates the extreme holdup problem in terms of capital for the seller who ends up having the marginal benefit of more capital for production in the DM exactly offset by the marginal cost of increased production. If we set $\theta = 0$, the capital holdup problem disappears as *ex-post* sellers can expropriate all the GNB surplus. However, in this case the buyer's money holdup problem is extreme. Thus there is no θ in the GNB case which can eliminate all holdup frictions.

Note that the first order condition (76) defines an implicit function of the solution $q = q(m, k_s, \hat{\mathbf{s}})$. That is q depends only on the money holding of the buyer and the DM-specific capital stock of the seller. This result is identical to Aruoba, Waller, and Wright [2009]. Therefore, we have the following everywhere (q, k_s)-smooth partial derivatives:

$$g_q := \frac{u_q(c_q/z)\left[\theta u_q + (1-\theta)c_q/z\right] + \theta(1-\theta)(u-c)[u_q(c_{qq}/z^2) - c_q u_{qq}/z]}{\left[\theta u_q + (1-\theta)c_q/z\right]^2} > 0, \tag{77}$$

and

$$g_k := \frac{u_q c_k\left[\theta u_q + (1-\theta)c_q/z\right] + \theta(1-\theta)(u-c)u_q c_{qk}/z}{\left[\theta u_q + (1-\theta)c_q/z\right]^2} < 0. \tag{78}$$

Moreover, since $u \in \mathbf{C}^2(\mathbb{R}_+)$ and $c \in \mathbf{C}^2(\mathbb{R}^2_+)$, by the Implicit Function Theorem, this implies that $q \in \mathbf{C}^1(\mathbb{R}^2_+)$. Specifically, we can sign the following partial derivatives:

$$\begin{aligned} &\frac{\partial d}{\partial m} = 1, \qquad \frac{\partial d}{\partial m_s,} = 0, \qquad \frac{\partial q}{\partial m} = \frac{A\phi}{w}\frac{1}{g_q} > 0, \\ &\frac{\partial q}{\partial m_s,} = 0, \qquad \frac{\partial d}{\partial k} = \frac{\partial m_s}{\partial k} = 0, \qquad \frac{\partial q}{\partial k} = -\frac{g_k}{g_q} > 0. \end{aligned} \tag{79}$$

D.1.2. *DM credit trades.* Assuming the buyer in these events has the same bargaining power θ, the outcome under monitored trades will be characterized by a first best allocation and a loan schedule, respectively, as

$$u_q(\breve{q}) = c_q(\breve{q}/z, k_s)/z,$$

and

$$\frac{A\phi}{w} l = (1-\theta)u(\breve{q}) + \theta c(\breve{q}, k_s, z) \equiv \breve{g}(\breve{q}, k_s).$$

D.1.3. *Envelope conditions.* At an optimum, the envelope conditions are as follows. The marginal value of money simplifies to

$$V_m(m, b, k, 0, \hat{\mathbf{s}}) = \frac{A\phi}{w}\left[\sigma\kappa\frac{u_q(q)}{g_q(q,k)} + (1-\sigma\kappa)\right] > 0, \tag{80}$$

where now g_q is defined in (77).

The DM marginal value of the capital stock above simplify to

$$\begin{aligned} V_k(m, b, k, 0, \hat{\mathbf{s}}) = &\frac{A}{w}(1+r) - \sigma\kappa\left[c_q(q_s/z, k)z^{-1}\frac{\partial q_s}{\partial k} + c_k(q_s/z, k)\right] \\ &- \sigma(1-\kappa)\left[c_q(\breve{q}_s/z, k)z^{-1}\frac{\partial \breve{q}_s}{\partial k} + c_k(\breve{q}_s/z, k) - \frac{A\phi}{w}\frac{\partial l_s}{\partial k}\right] \\ = &\frac{A}{w}(1+r) - \sigma\kappa\gamma(q, k, z) - \sigma(1-\kappa)(1-\theta)\left[\frac{(1-\theta)u_q(\breve{q})}{g_q(\breve{q}, k, z)}\right] c_k(\breve{q}/z, k). \end{aligned}$$

where

$$\gamma(q,k,z) = -c_q(q/z,k)\frac{1}{z}\frac{g_k(q,k,z)}{g_q(q,k,z)} + c_k(q/z,k) < 0.$$

APPENDIX E. NONSTOCHASTIC STEADY STATES AND CALIBRATIONS

In this section we outline how we calibrate the models. We consider first the baseline model with DM price taking. In section E.1, we discuss the model's definition of output from each sector and the resulting aggregate output for a country. Then we outline the steady state calculations for the baseline model in section E.2. In section E.3, we discuss the differences in the steady state conditions and an additional calibration target in terms of an aggregate markup of price over marginal cost.

E.1. **Measuring output.** For each country, the CM total (production) output in units of the final CM good, is

$$Y_{CM} = \hat{\phi}\hat{P}_h zF(K,H).$$

The DM total nominal output is $\sigma\kappa M + \sigma(1-\kappa)l$. Total real output in the DM, using ϕ^{-1} as the unit of account is

$$\begin{aligned} Y_{DM} &= \sigma\kappa\hat{M}\hat{\phi} + \sigma(1-\kappa)\hat{l}\hat{\phi} \\ &= \sigma\frac{(1-\tau_H)}{A}\left[\hat{\phi}\hat{P}_h zF_H(K,H)\right]\left[\kappa g(q,K,z) + (1-\kappa)\breve{g}(\breve{q},K,z)\right], \end{aligned}$$

where $g(q,K,z)$ is defined accordingly for each case, and

$$\breve{g}(\breve{q},K,z) = \begin{cases} \breve{q}\cdot c_q(\breve{q},K,z) & \text{if Price Taking} \\ (1-\theta)u(\breve{q}) + \theta c(\breve{q},K,z) & \text{if GNB.} \end{cases}$$

Total output, measured in terms of the CM final goods is:

$$\tilde{Y} = Y_{CM} + Y_{DM}.$$

Note that total output in terms of our aggregate DM and CM index good will be

$$Y = \frac{\hat{\phi}^{-1}Y_{CM} + p_{DM}Y_{DM}}{P_Y}.$$

E.2. **Baseline nonstochastic steady state characterizations.** From the stationary equilibrium demand for intermediate goods we have at steady state:

$$\hat{\phi}\hat{P}_h = G_{y_h}(y_h, y_f) := \left(\vartheta y_h^{\frac{1-\epsilon}{\epsilon}}\right)\left[G(y_h, y_f)\right]^{\frac{\epsilon-1}{\epsilon}}, \tag{81}$$

$$\hat{\phi}\hat{P}_f = G_{y_f}(y_h, y_f) := \left((1-\vartheta) y_f^{\frac{1-\epsilon}{\epsilon}}\right)\left[G(y_h, y_f)\right]^{\frac{\epsilon-1}{\epsilon}}, \tag{82}$$

$$\frac{\hat{\phi}^*\hat{P}_h}{\hat{e}} = G_{y_h^*}(y_h^*, y_f^*) := \left((1-\vartheta)(y_h^*)^{\frac{1-\epsilon}{\epsilon}}\right)\left[G(y_h^*, y_f^*)\right]^{\frac{\epsilon-1}{\epsilon}}, \tag{83}$$

$$\frac{\hat{\phi}^*\hat{P}_f}{\hat{e}} = G_{y_f^*}(y_h^*, y_f^*) := \left((1-\vartheta)(y_f^*)^{\frac{1-\epsilon}{\epsilon}}\right)\left[G(y_h^*, y_f^*)\right]^{\frac{\epsilon-1}{\epsilon}}. \tag{84}$$

The law of one price holds for intermediate goods, so that equating (81) and (83), we have

$$y_f = \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{\epsilon}{1-\epsilon}} y_h. \tag{85}$$

Using (86) in the aggregator G, we have

$$G(y_f, y_f) := \left[\vartheta y_h^{\frac{1}{\epsilon}} + (1-\vartheta) y_f^{\frac{1}{\epsilon}}\right]^{\epsilon} = \omega_I y_h, \tag{86}$$

where

$$\omega_I := \left[\vartheta + (1-\vartheta)\left(\frac{\vartheta}{1-\vartheta}\right)^{1/(1-\epsilon)}\right]^{\epsilon}.$$

From market clearing for Home-produced intermediate goods, we have

$$zK^{\alpha}H^{1-\alpha} = y_h + y_h^* \equiv \omega_F y_h, \tag{87}$$

where

$$\omega_F := \left[1 + \left(\frac{\vartheta}{1-\vartheta}\right)^{\epsilon/(1-\epsilon)}\right].$$

The resource constraint is

$$G(y_h, y_f) = (1+\tau_X)X + \delta K + \tau_H wH + \tau_K rK \equiv \omega_I y_h. \tag{88}$$

Equating (88) and (87) in terms of y_h, we have a relationship between CM production and final demand:

$$\left(\frac{\omega_I}{\omega_F}\right) zK^{\alpha}H^{1-\alpha} = (1+\tau_X)X + \left[(1-\alpha)\tau_H + \alpha\tau_K\right]\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} zK^{\alpha}H^{1-\alpha} + (1-\tau_K)\delta K.$$

Now dividing the above expression by H and defining $\Bbbk := K/H$, we obtain

$$X = \frac{1}{1+\tau_X}\left\{\left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\,\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}\right] z\Bbbk^{\alpha} - (1-\tau_K)\delta\Bbbk\right\} H. \tag{89}$$

Also, from the labor market clearing condition in the CM, we have, after evaluating U_X using the CRRA functional form indexed by parameters (B, γ):

$$X = \left[\frac{(1-\tau_H)(1-\alpha)B\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}}{A(1+\tau_X)} z\Bbbk^{\alpha}\right]^{1/\gamma}. \tag{90}$$

E.2.1. *Other side equations.* The following relations will be used in various other equations pinning down calibrations below. First, from (89), we can divide through by K to re-write as

$$\frac{X}{K} = \frac{1}{1+\tau_X}\left\{\left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\,\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}\right] z\Bbbk^{\alpha-1} - (1-\tau_K)\delta\right\}. \tag{89.a}$$

Further substitution of X out using (90) yields a relation between K and $\Bbbk$:

$$K = \frac{\left[\frac{(1-\tau_H)(1-\alpha)B\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}}{A(1+\tau_X)} z\Bbbk^{\alpha}\right]^{1/\gamma}}{\frac{1}{1+\tau_X}\left\{\left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\,\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}\right] z\Bbbk^{\alpha-1} - (1-\tau_K)\delta\right\}}. \tag{91}$$

From the DM credit trade outcomes we have $u_q(\breve{q}) = c_q(\breve{q}, K)$. Given the the parameterization of u and c, indexed by parameters (C, η) and ϖ, respectively, we then have

$$\breve{q} = \left(\frac{C}{\varpi}\right)^{\frac{1}{\eta+\varpi-1}} K^{\frac{\varpi-1}{\eta+\varpi-1}}. \tag{92}$$

From the Euler equation for money holdings at steady state we also gets

$$\frac{1}{\sigma\kappa}\left[\beta^{-1} - (1-\sigma\kappa)\right] c_q(q, K) = u_q(q),$$

where there is a wedge $(\sigma\kappa)^{-1}\left[\beta^{-1} - (1-\sigma\kappa)\right]$ arising from matching frictions, relative to a first-best chracterization for the allocation of q. Using the parameterization of u and c, we have explicitly a relation between q and K:

$$q = \left(\frac{\sigma\kappa \cdot C}{\varpi\left[\beta^{-1} - (1-\sigma\kappa)\right]}\right)^{\frac{1}{\eta+\varpi-1}} K^{\frac{\varpi-1}{\eta+\varpi-1}}. \tag{93}$$

From the Euler equation for capital, we have the steady state relation between $\Bbbk$ and K:

$$\delta = \frac{1-\beta^{-1}}{1-\tau_K} + (\theta\omega_I^{\frac{\epsilon-1}{\epsilon}})\alpha z\Bbbk^{\alpha-1} - \frac{\sigma(1+\tau_X)}{(1-\tau_K)U_X(X)}[\kappa\gamma(q, K) + (1-\kappa)\gamma(\breve{q}, K)]. \tag{94}$$

E.2.2. *Calibrating A.* Equating (89) and (90), we get an expression that allows us to calibrate (given target H along with other parameters), the marginal disutility of labor in the CM:

$$A = \left[(1+\tau_X)^{-1}(1-\tau_H)(1-\alpha)B\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} z\mathbb{k}^{\alpha}\right]\left(\frac{1+\tau_X}{H}\right)^{\gamma} \times \left\{\frac{1}{\left[\frac{\omega_I}{\omega_F} - \left((1-\alpha)\tau_H + \alpha\tau_K\right)\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}\right] z\mathbb{k}^{\alpha} - (1-\tau_K)\delta\mathbb{k}}\right\}^{\gamma}. \tag{95}$$

E.2.3. *Calibrating ϖ.* From our definition of real output $\tilde{Y}$ in terms of the CM final good as numeraire, we have

$$\tilde{Y} = \hat{\phi}\hat{P}_h zF(K,H) + \sigma\frac{(1-\tau_H)}{A}\left[\hat{\phi}\hat{P}_h zF_H(K,H)\right]\left[\kappa g(q,K) + (1-\kappa)\breve{g}(\breve{q},K)\right].$$

Divide both sides by K, knowing that $\hat{\phi}\hat{P}_h = \vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}$. Then we have a relation between another calibration target, the output-to-capital ratio $s_K^{-1} := Y/K$, and the capital complementarity parameter ϖ, given other calibrations:

$$s_K^{-1} = \vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} z\mathbb{k}^{\alpha-1}\left\{1 + \frac{\sigma(1-\tau_H)\varpi}{A}\left[\kappa q^{\varpi} + (1-\kappa)\breve{q}^{\varpi}\right]K^{-\varpi}\mathbb{k}\right\}. \tag{96}$$

E.2.4. *Calibrating α.* Following Aruoba, Waller, and Wright [2009], we calibrate α to match the labor share of CM output, denoted as LS. Assuming the Cobb-Douglas parameterization of F, we have the relations

$$\alpha = -\left(\frac{\ln(z\cdot LS)}{\ln(K) - \ln(H)}\right). \tag{97}$$

E.2.5. *Calibrating σ.* In this section, we describe how we derive the calibration target variable – the nominal-interest-rate semi-elasticity of money demand, ξ – in the search-theoretic models. This target is used for calibrating the value of σ.

The steps below apply to all three types of decentralized market (DM) pricing mechanism assumptions, with the appropriate definitions for partial derivatives. Computationally, these are modular objects that are easily applied. These steps are similar to Aruoba, Waller, and Wright [2009] with the exception that we now have to account for traded goods relative prices as well.

Consider a generic equilibrium pricing condition for trades involving money in the DM:

$$\frac{A\phi M}{w} = g(q,K,z),$$

where we had defined $w := \tilde{w}(1-\tau_H) = (1-\tau_H)\phi P_h zF_H(K,H)$ in the paper, and, $(q,K)\mapsto g(q,K)$ depends on the pricing mechanism assumed.

Step 1. The nominal interest rate (i) elasticity of real money demand (ϕM) is defined by

$$\xi = \frac{\partial(\phi M)}{\partial i} \cdot \frac{i}{\phi M}. \tag{98}$$

Using the DM pricing condition, we have

$$\xi = \left[g_q(q,K,z) \cdot \frac{\partial q}{\partial i} + g_K(q,K,z) \cdot \frac{\partial K}{\partial i} \right] \frac{i}{g(q,K,z)} + z \left[F_{HH}(K,H) \cdot \frac{\partial H}{\partial i} + F_{HK}(K,H) \cdot \frac{\partial K}{\partial i} \right] \frac{i}{F_H(K,H)} + \frac{\partial \hat{\phi}\hat{P}_h}{\partial i} \cdot \frac{i}{\hat{\phi}\hat{P}_h}.$$

Proposition 1. *In a cross-country symmetric steady state, the last term,*

$$\frac{\partial \hat{\phi}\hat{P}_h}{\partial i} \cdot \frac{i}{\hat{\phi}\hat{P}_h} = 0.$$

Proof. Observed that in a steady state,

$$\hat{\phi}\hat{P}_h = G_{y_h}(y_h, y_f) = \vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}},$$

where $\omega_I := [\vartheta + (1-\vartheta)(\vartheta/(1-\vartheta))^{1/(1-\epsilon)}]^{\epsilon}$, is just a constant. □

Hence we only need three independent conditions to pin down the partial derivatives: $\partial q/\partial i$, $\partial H/\partial i$, and $\partial K/\partial i$.

Step 2. In a deterministic steady state, we have the following conditions arising from the money Euler equation, capital Euler equation and the resource constraint:

$$i = \sigma\kappa \left[\frac{u_q(q)}{g_q(q,K,z)} - 1 \right],$$

$$i = \left[\vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}} \cdot zF_K(K,H) - \delta \right] (1-\tau_K) - \sigma(1+\tau_X) \left[\frac{(1-\tau_H)\vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}}}{A(1+\tau} \right] zF_H(K,H) \times \left[\kappa\gamma(q,K,z) + (1-\kappa)\check{\gamma}(\check{q},\check{K},z) \right],$$

$$X = \frac{1}{1+\tau_X} \left\{ \left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\, \vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} \right] zF(K,H) - \delta(1-\tau_K)K \right\},$$

where $i \equiv \beta^{-1} - 1$, γ is defined according in each of the DM competitive price taking or bargaining cases,

$$\check{\gamma}(\check{q},\check{K},z) = c_k(q/z,k)$$

in the case of DM competitive price taking, and,

$$\check{\gamma}(\check{q},\check{K},z) = (1-\theta) \left(\frac{(1-\theta)u_q(q)}{g_q(q,k,z)} \right) c_k(q/z,k)$$

in the case of DM bargaining. Note that we can solve for $\breve{q}$ from the first-best allocation under credit trades

$$u_q(\breve{q}) = c_q(\breve{q}, K) = 0.$$

Thus we also know that

$$\frac{\partial \breve{q}}{\partial K} = \frac{c_{qK}(\breve{q}, K)}{u_{qq}(\breve{q}) - c_{qq}(\breve{q}, K)},$$

which will be utilized in the next step.

Step 3. Take the total derivative of the system in Step 2, to obtain the following system of equations:

$$1 \cdot di = m_{11} \cdot dq + m_{12} \cdot dK + m_{13} \cdot dH,$$

$$1 \cdot di = m_{21} \cdot dq + [m_{22}^1 + m_{22}^2 + m_{22}^3] \cdot dK + [m_{23}^1 + m_{23}^2] \cdot dH$$

$$0 \cdot di = m_{31} \cdot dq + [m_{32}^1 + m_{32}^2] \cdot dK + [m_{33}^1 + m_{33}^2] \cdot dH,$$

where

$$m_{11} := \sigma\kappa \left[g_q(q, K, z) u_{qq}(q) - u_q(q) g_{qq}(q, K, z) \right],$$

$$m_{12} := -\sigma\kappa \left[u_q(q) g_{qK}(q, K, z) / (g_q(q, K, z))^2 \right],$$

$$m_{13} := 0;$$

and,

$$m_{21} := -\frac{\sigma\kappa(1 + \tau_X \gamma_q(q, K, z)}{U_X(X)},$$

$$m_{22}^1 := \vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}} z F_{KK}(K, H)(1 - \tau_K) - \frac{\sigma(1 + \tau_X)}{U_X(X)} \kappa \gamma_K(q, K, z),$$

$$m_{22}^2 := \frac{\sigma(1 + \tau_X) U_{XX}(X)}{[U_X(X)]^2} \left[\frac{(1 - \tau_H)\vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}}}{A(1 + \tau_X)} z F_{KK}(K, H) \right] [\kappa\gamma(q, K, z) + (1 - \kappa)\breve{\gamma}(\breve{q}, \breve{K}, z)],$$

$$m_{22}^3 := -\frac{\sigma(1 + \tau_X)}{U_X(X)} (1 - \kappa) \breve{\gamma}_K(\breve{q}, \breve{K}, z);$$

and,

$$m_{23}^1 := \vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}} z F_{KH}(K, H)(1 - \tau_K),$$

$$m_{23}^2 := \frac{\sigma(1 + \tau_X) U_{XX}(X)}{[U_X(X)]^2} \left[\frac{(1 - \tau_H)\vartheta \omega_I^{\frac{\epsilon-1}{\epsilon}}}{A(1 + \tau_X)} z F_{KH}(K, H) \right] [\kappa\gamma(q, K, z) + (1 - \kappa)\breve{\gamma}(\breve{q}, \breve{K}, z)];$$

and,

$$m_{31} := 0,$$
$$m_{32}^1 := U_X(X) + zF_{HK}(K,H),$$
$$m_{32}^2 := zF_H(K,H) \cdot \frac{U_{XX}(X)}{1+\tau_X} \left\{ \left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\, \vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} \right] zF_K(K,H) - \delta(1-\tau_K) \right\},$$
$$m_{33}^1 := U_X(X) + zF_{HH}(K,H),$$
$$m_{33}^2 := (zF_H(K,H))^2 \cdot \frac{U_{XX}(X)}{1+\tau_X} \left\{ \left[\frac{\omega_I}{\omega_F} - ((1-\alpha)\tau_H + \alpha\tau_K)\, \vartheta\omega_I^{\frac{\epsilon-1}{\epsilon}} \right] \right\}.$$

This is a linear map written compactly as

$$\mathbf{M}_{(q,K,H)} \begin{pmatrix} dq/di \\ dK/di \\ dH/di \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We can thus find the projection under the map $\mathbf{M}_{(q,K,H)}$ from the point $(1,1,0)$ as

$$\begin{pmatrix} dq/di \\ dK/di \\ dH/di \end{pmatrix} = \mathbf{M}_{(q,K,H)}^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Step 3. Given the steady state values for (q, K, H) we can now solve for the value of ξ in Step 1.

E.2.6. *Calibrating B.* In Aruoba, Waller, and Wright [2009], the authors calibrate B to match a measure of money demand elasticity. In our setting, since the DM sector also behaves like a nontraded goods sector where both money and credit are used, we choose to calibrate B to a calibration target of the nontraded-goods consumption share. In our model this is just the DM consumption to total consumption ratio:

$$NTS = \frac{Y_{DM}}{X + Y_{DM}}. \tag{99}$$

□ *Calibration summary.* Along with (90), (91), (92) and (93), we have a system in (94), (95), (96), (97), (98) and (99), characterizing the solutions $(A, \alpha, B, \varpi, \sigma, \Bbbk)$. We minimize a quadratic loss criterion in terms of deviations from the targets $(H, LS, NTS, K/Y, v)$ subject to the system of nonlinear equations (94), (95), (96), (97), (98) and (99).

E.3. **GNB and nonstochastic steady state characterizations.** The only difference in the characterization of steady state allocations now appears in terms of the determination of steady state $(q, \Bbbk)$ where $\Bbbk := K/H$ is the capital-labor ratio. Specifically, from the Euler equation for money at steady state, we can derive a relation between q and K at steady state, assuming the functional forms for preferences and technology as in the baseline model's example:

$$\frac{1}{\sigma\kappa}[\beta^{-1} - (1-\sigma\kappa)]g_q(q,K,z) = u_q(q).$$

Now, with GNB, the g_q function involves second-order derivative functions of u and c, so that the relation above cannot be explicitly written in terms of a exact relation between q and K. Nevertheless, we can find the steady state points numerically.

Likewise, from the Euler equation characterizing equilibrium capital accumulation, we can derive a steady state relation solving implicitly for $\mathbb{k}$ as:

$$\begin{aligned}\delta =& \frac{1-\beta^{-1}}{1-\tau_K} + (\theta\omega_I^{\frac{\epsilon-1}{\epsilon}})\alpha z\mathbb{k}^{\alpha-1} \\ &- \frac{\sigma(1+\tau_X)}{(1-\tau_K)U_X(X)}\left\{\kappa\gamma(q,K) + (1-\kappa)(1-\theta)\left[\frac{(1-\theta)u_q(\breve{q})}{g_q(\breve{q},K,z)}\right]c_K(\breve{q}/z,K)\right\}.\end{aligned}$$

where $\omega_I := [\vartheta + (1-\vartheta)(\vartheta/(1-\vartheta))^{1/(1-\epsilon)}]^{\epsilon}$, and, where

$$\gamma(q,K,z) = -\frac{1}{z}c_q(q/z,K)\frac{g_K(q,K,z)}{g_q(q,K,z)} + c_K(q/z,K) < 0.$$

Note that we know X and K can be written analytically as functions of $\mathbb{k}$, exactly, as in the baseline model.

E.3.1. *Calibrating θ.* The markup μ_M in monetary trades in the DM satisfies the definition

$$1+\mu_M = \frac{M/q}{\frac{c_q(q/z,K)}{zA\phi/w}} = \frac{g(q,K,z)}{qc_q(q/z,K)/z},$$

where $g(q,K,z)$ is now defined by (76).

The markup in credit trades μ_l satisfies

$$1+\mu_l = \frac{l/\breve{q}}{\frac{c_q(\breve{q}/z,K)}{zA\phi/w}} = \frac{\breve{g}(\breve{q},K,z)}{\breve{q}c_q(\breve{q}/z,K)/z}.$$

where $\breve{g}(\breve{q},K,z) := (1-\theta)u(\breve{q}) + \theta c(\breve{q}/z,K)$.

So average markup coming from the DM is still $\mu_{DM} = \kappa\mu_M + (1-\kappa)\mu_l$. The aggregate markup is $\mu := (Y_{DM}/Y)\mu_{DM} + (Y_{CM}/Y)\cdot 0$, where $Y = Y_{CM} + Y_{DM}$.