WORKING PAPERS IN ECONOMICS & ECONOMETRICS

A New Approach to Solving Production-Appropriation Games with Many Heterogeneous Players

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May 14, 2010

Abstract

We set out a model of production and appropriation involving many players, who differ with respect to both resource endowments and productivities. We write down the model in a novel way that permits our analysis to avoid the proliferation of dimensions associated with the best response function approach as the number of heterogeneous players increases. We establish existence of a unique equilibrium in pure strategies, and carry out some comparative static exercises.

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1 Introduction

Much attention has been devoted recently to developing and exploring formal models in which individuals or groups may find it worthwhile to devote part, or even all, of their resources to the appropriation of the hard-won output of others rather than to the production of net output. Edgeworth [7] referred to the use of conflict, or war, as a significant alternative resource allocation device. Pareto, too, noted the significance of appropriation as a means of acquiring economic goods[1]. Haavelmo [12] and Skogh and Stewart [14] provided interesting formal models of appropriation, but it is the more recent work by Hirshleifer and Grossman that seems to have fired the imagination of other scholars and provoked them into further analysis of these matters: see, in particular, Hirshleifer [13], and Grossman and Kim [11]. Most recently, Anderton and Carter [1] have provided a useful book-length survey of the rapidly burgeoning literature on conflict economics.

Various treatments of these issues may be found in the collection of papers edited by Garfinkel and Skaperdas [8]. Garfinkel and Skaperdas [9] provide a valuable exposition of a simple model of production and appropriation. Like much of the extant literature, which they have recently surveyed more broadly in [10], their formal exposition focuses on a simple two-player model. Indeed, much of the recent literature restricts attention to two-player models. Alternatively, authors either assume many identical players and a symmetric equilibrium, or resort to numerical simulation. To our knowledge, there does not currently exist an analytical demonstration of the existence of a unique Nash equilibrium in pure strategies in the presence of many heterogeneous players. All three approaches raise concerns about the robustness of conclusions reached, and the scope of the analysis in terms of the kinds of questions that it can address.

We suggest an alternative way of writing down models of conflict and appropriation. We build on the simple two-player model of Garfinkel and Skaperdas [10]. We incorporate the possibility that each player automatically has access to a certain proportion of her own output as a consumer without having to contest that portion, and also a private cost associated with appropriative activity on the part of each player that is in addition to its opportunity cost. Most significant, however, is the ability of our approach to provide a general qualitative analysis of situations involving potentially large numbers of players who may differ with respect to resource endowments, productivities as producers, and the cost of thieving. We solve the resulting model using an extension of a novel technique of analysis recently used by Cornes and Hartley [2] to analyze other games with aggregative structure. Dickson and Hartley [5] have recently extended and applied this approach to analyze market games as examples of noncooperative games in which two aggregates may be regarded as providing sufficient statistics to describe an allocation, regardless of how many heterogeneous players take part. In their analysis, the two aggregates are the total quantities of the two goods traded. The present application is a little more intricate, but similarly describes an allocation in terms of just two aggregates, regardless of the number of heterogeneous players. We

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[1]Note Pareto's observation, translated by Hirshleifer [13], that

> "the efforts of men are utilized in two different ways: they are directed to the production or transformation of economic goods, or else to the appropriation of goods produced by others."

establish the existence of a unique equilibrium in pure strategies, and we present some illustrative comparative static exercises. Finally, we suggest new questions that may be explored in models of production and appropriation in the presence of three or more heterogeneous players. One characteristic of our analysis—which we regard as a virtue—will become evident as we proceed. This is its susceptibility to a relatively simple geometric representation, which can provide a more intuitive understanding of what is going on and why.

The model of production and appropriation in this paper is a simple one. Our aim is primarily to explore an alternative way of writing down such models that promises a more powerful and insightful way of modeling situations involving many heterogeneous players. Our concluding comments suggest further extensions of the model that broaden its scope while retaining its aggregative character. In addition, we draw attention to the potential application of our technique to other economic problems involving large numbers of heterogeneous players.

2   An n-player model of production and appropriation

There are n players, and the set of all players is I. Player i's exogenous endowment, e_i, is divided between two activities: production, and appropriation. We denote the level of i's productive activity by x_i. Player i's allocation to appropriative activity is g_i. Player i's constraint set requires that $e_i = x_i + g_i$. We will denote the aggregate level of appropriative activity by $G = \sum_{j=1}^{n} g_j$.

We denote the output of player i by $y_i = a_i x_i$, where a_i is an exogenous productivity parameter, and $Y = \sum_{j=1}^{n} a_j x_j$. An exogenously given proportion, λ, of player i's output is available for consumption by that player[2]: $0 \leq \lambda < 1$. The remainder goes into a common pool, and players may devote resources to expropriation from this pool. The share of the common pool that is enjoyed as consumption by player i reflects her share of total effort devoted to appropriation, g_i/G. Her total level of consumption is given by

$$z_i = \lambda y_i + \frac{g_i}{G} \left(1 - \lambda\right) Y. \tag{1}$$

We assume that player i's utility is a linear function of her consumption and the effort that she devotes to appropriation:

$$u_i \left(z_i, g_i\right) = z_i - b_i g_i$$

where the parameter b_i captures the cost to player i per unit of appropriation. A player is fully described by the trio (e_i, a_i, b_i). Using (1) to substitute for z_i, player i's payoff can be written as

$$v_i \left(g_i, G, Y\right) = \lambda a_i \left(e_i - g_i\right) + \frac{g_i}{G} \left(1 - \lambda\right) Y - b_i g_i. \tag{2}$$

We follow the existing literature in exploring the properties of pure strategy Nash noncooperative equilibrium. Equation (2) expresses player i's payoff as a function of her own

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[2]Note that we exclude the possibility that $\lambda = 1$. In the event of $\lambda = 1$, reflecting the fact that every player's output is automatically available to that player as consumption, there is simply no place for appropriation.

action and two aggregate quantities. The conventional way of expressing i's payoff in the noncooperative game context would be

$$\pi_i \left(g_i, G_{-i}, Y_{-i}\right) = \lambda a_i \left(e_i - g_i\right) + \frac{g_i}{G_{-i} + g_i} \left(1 - \lambda\right) \left(Y_{-i} + a_i \left(e_i - g_i\right)\right) - b_i g_i$$

where $G_{-i} \equiv G - g_i$ and $Y_{-i} \equiv Y - y_{-i}$. The quantities G_{-i} and Y_{-i} are parametrically fixed when describing player i's behavior in the noncooperative game, and they would appear as arguments in that player's best response function.

One interpretation of λ sees it as reflecting the extent to which the state provides security to individuals by ensuring their private property rights over some part of their own output. A situation in which $\lambda = 0$ reflects an anarchic world in which no individual's output is safe from predation by others. A strictly positive value of λ reflects the existence of institutions that protect a proportion of an individual's output from predation by others.

An interesting alternative interpretation turns the role of the state on its head. Here, the parameter λ may be interpreted as reflecting the presence of a confiscatory state, which taxes a fraction of individual output. Players then devote resources to conflict, maybe in the form of lobbying, in order to acquire for their own consumption some portion of the confiscated output. Our model draws attention to an efficiency cost associated with taxation, in the form of the socially unproductive conflict that it generates through encouraging the contest among players to claw back the resources that the government has expropriated. This is distinct from the standard deadweight loss, from which we abstract, associated with distortionary taxation. Garfinkel and Skaperdas [10], in their recent long survey, interpret g_i/G as the probability of i's winning the output in the common pool, rather than the proportion of that output that is acquired with certainty. Either interpretation is admissible given risk neutrality on the part of all players.

An alternative interpretation of the payoff function is suggested by the observation that, if total output Y were exogenously given and λ were zero, i's payoff function would have precisely the same form as that encountered in the theory of contests involving risk neutral players. Hence our formulation extends contest theory by allowing the prize to depend endogenously on choices made by the individual contestants. Each player chooses how much effort to devote to increasing the total size of the pie, and how much to devote to increasing her share of that pie.

Finally, the parameter b may be interpreted as reflecting the degree to which prevailing norms engender a feeling of guilt, or attitude of opprobrium, associated with the act of appropriating the output of others. An alternative interpretation is that, rather than being motivated from within, players are discouraged from stealing by policies that influence the probability of detection and the severity of punishment in the event of detection. Whatever the interpretation, one can pose questions about whether policies or campaigns to increase the value of b may make individuals in society better off, and indeed about the socially optimal value of such a parameter.

3   Modeling player i's behavior

We are interested in Nash noncooperative equilibria in pure strategies of the game in which, for all $i = 1, ..., n$, player i chooses $g_i \in [0, e_i]$ to maximize her payoff. The first-order

condition characterizing an interior solution for player i is

$$\frac{\partial \pi_i}{\partial g_i} = \frac{\partial v_i}{\partial g_i} + \frac{\partial v_i}{\partial G}\frac{\partial G}{\partial g_i} + \frac{\partial v_i}{\partial Y}\frac{\partial Y}{\partial g_i} = 0. \tag{3}$$

We denote the level of g that satisfies this first-order condition by $\widetilde{g}_i$.

$$\widetilde{g}_i = \frac{\left[(1-\lambda)\,Y - (\lambda a_i + b_i)\,G\right]G}{(1-\lambda)\,(Y + a_i G)}$$

We rewrite this more economically as

$$\widetilde{g}_i = \left(\frac{Y - c_i G}{Y + a_i G}\right)G, \tag{4}$$

where

$$c_i \equiv \frac{\lambda a_i + b_i}{1 - \lambda}. \tag{5}$$

Although we are primarily interested in the equilibrium values of the aggregates G and Y, it will prove convenient to undertake some of the analysis in terms of the variables G and $p \equiv Y/G$. We can write (4) as

$$\widetilde{g}_i = \left(\frac{p - c_i}{p + a_i}\right)G. \tag{6}$$

Description of player i's behavior must take into account the possibilities of corner solutions at which she chooses to allocate all of her endowment either to production or to appropriation. Denote player i's most preferred value of appropriation by $\widehat{g}_i$. Inspection of (6) confirms that, if $p - c_i < 0$, the non-negativity constraint on $\widehat{g}_i$ is binding. Conversely, if the expression (6) exceeds e_i, then player i will devote all her resources to appropriation. It may be readily confirmed that, if an interior solution exists, it satisfies the second order condition associated with a payoff maximum.

A complete description of player i's most preferred level of appropriation is given by

$$\widehat{g}_i = \rho_i(p, G) = \begin{cases} 0 & \text{if } \widetilde{g}_i < 0 \\ \widetilde{g}_i & \text{if } \widetilde{g}_i \in [0, e_i] \\ e_i & \text{if } \widetilde{g}_i > e_i. \end{cases} \tag{7}$$

At any given allocation, depending on a player's parameter values, she will belong to one of three groups—she may choose only to produce output, she may both produce and appropriate from others, or she may only appropriate output. Figure 1 shows, in (p, G) space, into which group player i falls in terms of the prevailing values of the aggregate variables. It is worth drawing attention to some significant properties of this diagram. First, the boundary between allocations at which $\widehat{g}_i = 0$ and those at which $0 < \widehat{g}_i < e_i$, labeled i, is linear—indeed, it is parallel to the G axis. Second, the boundary between the allocations at which $\widehat{g}_i = e_i$ and those at which $0 < \widehat{g}_i < e_i$, labeled i^+, describes, with appropriate change of variables, the positive section of a rectangular hyperbola[3]. Its asymptotes are the vertical line along which

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[3]This may be seen by noting that the condition that describes the boundary, $\widetilde{g}_i = e_i$, may be rewritten in the form $(p - c_i)\,(G - e_i) = k_i \equiv e_i\,(a_i + c_i)$: the product of the variables $(p - c_i)$ and $(G - e_i)$ equals a constant, k_i.
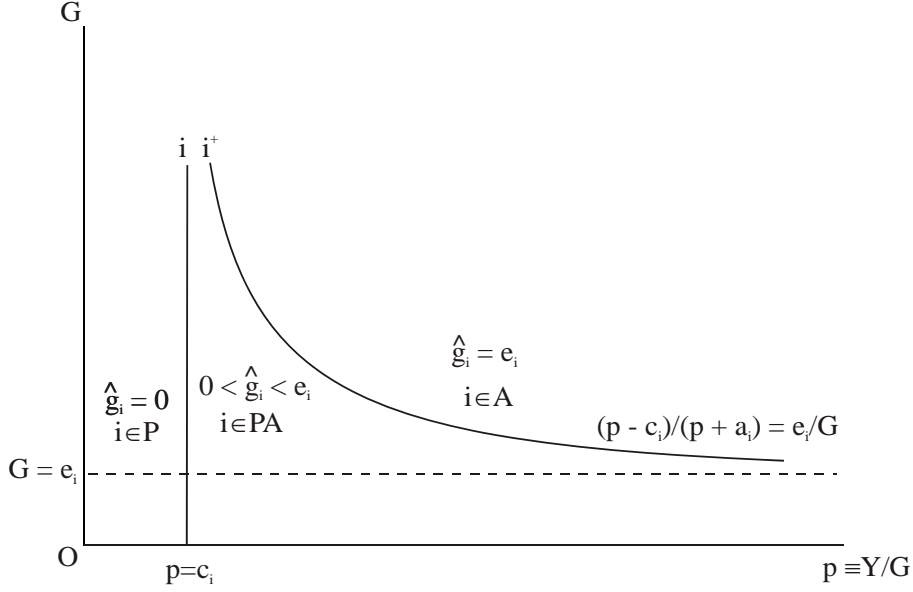
Figure 1:

$p = c_i$ and the horizontal dashed line along which $G = e_i$. Clearly, for every player, (p, G) space may be divided up into the three regions, according to which group the player belongs to at the prevailing values of p and G. Insofar as players' endowments, productivities or aversions to thieving differ, the precise locations of these regions will vary across players. But, for each player, the qualitative features of the boundaries are as described above and shown in Figure 1.

At any given allocation, the set I of all players can be decomposed into three mutually exclusive and exhaustive subsets: $I = P \cup PA \cup A$, where P is the set of full-time producers, PA that of producer-appropriators, and A that of full-time appropriators. For each player, we will refer to these subsets as regimes.

4 Nash equilibrium in the n-player model

It is important to emphasize again that our equilibrium concept is the standard Nash non-cooperative equilibrium in pure strategies. However, our approach permits a novel characterization of such an equilibrium in terms of just two consistency conditions in the two unknowns, p and G, fulfilment of which is necessary and sufficient for an allocation to be a Nash noncooperative equilibrium in pure strategies. Recall that, for every player, (7) uniquely determines the most preferred level of appropriation that is consistent with the observed pair, (p, G).

Our first consistency condition, that of "Appropriative Consistency", requires that the values of the aggregate variables be such that the sum of the implied most preferred individual appropriation levels equals the observed aggregate level of appropriation, G:

Appropriative Consistency. The sum of replacement values must equal the aggregate

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value of appropriation. Formally, this requires that any appropriatively consistent pair $\left(p^A, G^A\right)$ must satisfy the following relationship:

$$\sum_{i \in I} \rho_i(p^A, G^A) = G^A. \tag{8}$$

Given the partitioning of the players into the three groups—which itself is, of course, endogenous—this consistency requirement may be written as

$$\sum_{i \in PA} \frac{p^A - c_i}{p^A + a_i} + \sum_{i \in A} \frac{e_i}{G^A} = 1. \tag{9}$$

We will use this last formulation of the appropriative equilibrium condition in our subsequent demonstration of the existence of a unique equilibrium.

Our second condition, that of "Productive Consistency", requires that the values of the aggregate variables be such that the implied sum of most preferred individual levels of production should equal the observed aggregate output level, Y. Player i's most preferred level of production is $\widehat{y}_i = a_i\widehat{x}_i = a_i\left(e_i - \widehat{g}_i\right) = a_i\left(e_i - \rho_i\left(p, G\right)\right)$. Summing over all players and rearranging yields the following statement of productive consistency requirement:

Productive Consistency. The productivity-weighted sum of individual appropriation levels must equal that of total endowment less total output:

$$\sum_{i \in I} a_i \rho_i\left(p^P, G^P\right) = \sum_{i \in I} a_i e_i - Y^P. \tag{10}$$

This requirement may alternatively be written as

$$\sum_{i \in PA} a_i \left(\frac{p^P - c_i}{p^P + a_i}\right) + \sum_{i \in A} \frac{a_i e_i}{G^P} = \sum_{i \in I} \frac{a_i e_i}{G^P} - p^P. \tag{11}$$

Fulfilment of both (9) and (11) is necessary and sufficient for an allocation to be a Nash equilibrium. They provide a complete description of a Nash equilibrium in pure strategies as a solution to two equations in two unknowns, regardless of how many players there are and regardless of the extent to which these players may differ from one another in terms of endowments, productivities or costs of appropriation. We will indicate a Nash equilibrium allocation as (p^*, G^*).

We will now show that the properties of the two consistency conditions, (9) and (11), ensure that they possess a unique solution for the variables G and p. This immediately implies that there exists a unique Nash equilibrium in pure strategies. Our argument in the text will proceed by analyzing the two relationships described by (9) and (11). Having claimed as one virtue of our approach that it is susceptible to simple geometric representation, we will proceed geometrically by examining the graphs implied by the two consistency requirements. We will show that their shapes imply the existence of a single point of intersection. We begin by considering (9), the condition for appropriative consistency.
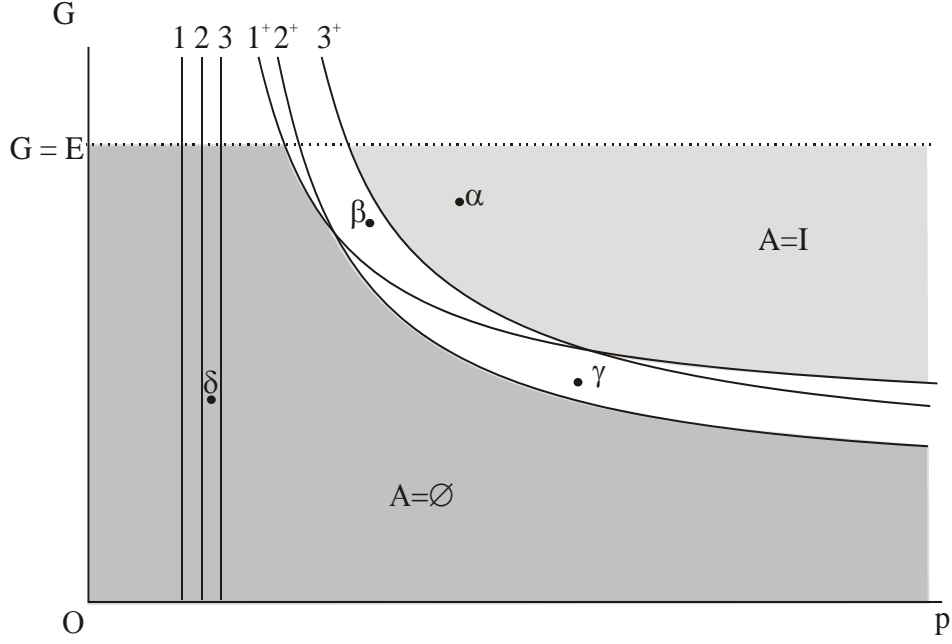
Figure 2:

4.1 Appropriative Consistency

Recall the appropriative consistency requirement, (9), where the assignment of players to the sets PA and A at any point in (p, G) space is itself endogenous. Figure 2 shows the boundaries between the three regimes for each player in a three-player example. The aggregate endowment of the economy implies that only allocations represented by points on or below the dotted line corresponding to $G = E$, where $E \equiv \sum_{j \in I} e_j$, are feasible.

The lines labeled $i \, (= 1, 2, 3)$ show the allocations at which player i is indifferent between pure production and production-appropriation. Those labeled i^+ show allocations at which player i is indifferent between production-appropriation and pure appropriation. The locations of each of these boundaries are determined by the exogenous parameters within the model.

These boundaries divide (p, G) space into a number of regions, each corresponding to a given assignment of all players to one of these three sets. For example, at the allocation represented by the point α—indeed, at all allocations within the shaded area marked $A = I$— all three players would choose to be pure appropriators. At β, player 3 would choose both to produce and to appropriate, while players 1 and 2 would choose pure appropriation. At γ, player 2 would choose pure appropriation while players 1 and 3 would choose to produce and appropriate. Finally, at δ player 3 would only produce, while players 1 and 2 would both produce and appropriate.

It may be worth drawing explicit attention to three types of region that are of particular interest.

  1. Regions within which $A = \varnothing$. If an appropriatively consistent allocation lies within

such a region, then in the neighborhood of any such allocation the requirement becomes

$$\sum_{i \in PA} \left(\frac{p^A - c_i}{p^A + a_i} \right) = 1.$$

Suppose, for a moment, that we are looking at an allocation within such a region that satisfies the appropriative consistency condition. Then the form taken by the condition uniquely determines the required value of p. Within this region, the graph consists of a vertical segment corresponding to the unique value of p that satisfies this condition for the relevant set of players.

2. Regions within which $PA = \varnothing$. If an appropriatively consistent allocation lies within a region in which $PA = \varnothing$, then in the neighborhood of any such allocation the requirement becomes

$$\sum_{i \in A} \frac{e_i}{G^A} = 1.$$

This uniquely determines the required value of G, and the graph within such a region is a horizontal line along which $G = \sum_{i \in A} e_i$. Note further that, if the set P is also empty, so that $A = I$, then the condition implies that $G = E$.

3. Regions within which both A and PA are non-empty. Within such a region, both of the summation terms appear on the left hand side of the consistency condition. Inspection makes it clear that, starting from an appropriatively consistent allocation in such a region, an increase in p raises the term $\sum_{i \in PA} \left(\frac{p^A - c_i}{p^A + a_i} \right)$ and must therefore be accompanied by an increase in the value of G in order to maintain appropriative consistency.

Armed with these observations, it is an easy matter to piece together the graph of allocations that satisfy the appropriative consistency requirement. First, consider the shaded set in the upper right-hand portion of Figure 3 where $A = I$. This must clearly be non-empty. Points within this set on the line along which $G = E$ certainly satisfy the appropriative consistency condition. Moreover, they are the only allocations within this set that do so. At such points, every player chooses to devote her entire endowment to appropriation, and the aggregate level of appropriation must equal the aggregate endowment. This set is represented by the thick horizontal line running from the point $(\overline{p}, E)$ to (∞, E).

At the point $(\overline{p}, E)$, player 3 is on the boundary between being a pure appropriator and a producer-appropriator. Now reduce p by a small amount below $\overline{p}$. Player 3 is no longer a pure appropriator, but is a producer-appropriator. Since the sets A and PA each now have at least one element, inspection of (9) shows that further decline in p must be accompanied by a decline in G to maintain appropriative consistency. The graph of appropriatively consistent allocations is strictly upward-sloping through this region. Eventually, it meets the boundary labeled 2^+. At this point, player 2 switches from pure appropriation to production-appropriation. The graph continues to be upward-sloping. At last, it meets the boundary labeled 1^+, at which player 1 switches to production-appropriation. At this point, the set A becomes empty and the appropriative consistency requirement becomes $\sum_{i \in PA} \frac{p^A - c_i}{p^A + a_i} = 1$, which uniquely determines the value of p at, say, $\underline{p}$.
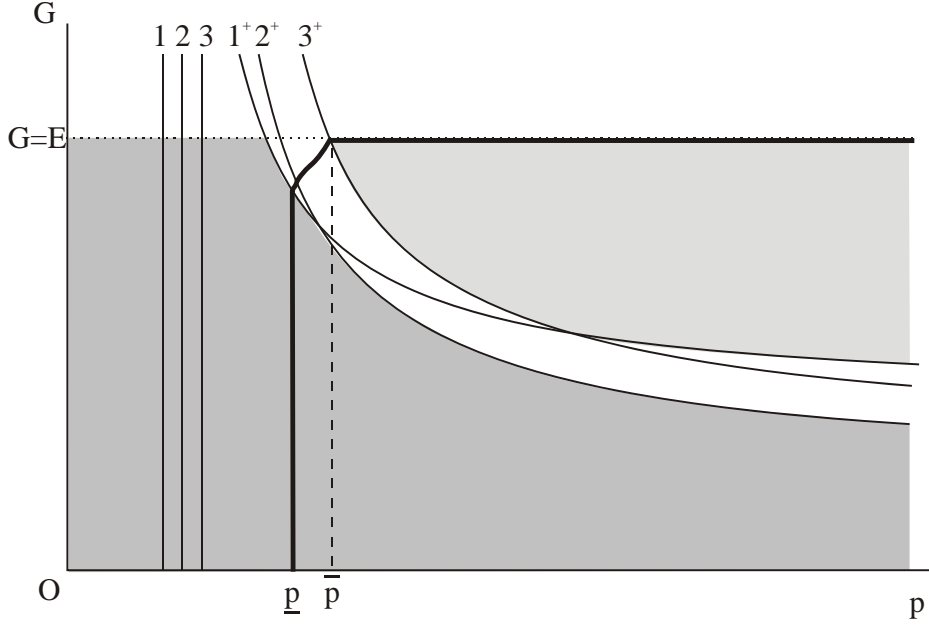
Figure 3:

Our claim will be that the graph of the appropriative consistency requirement, which is the thick line in Figure 3, has qualitative properties that, in combination with those that we will establish for the graph associated with productive consistency, are sufficient to ensure existence and uniqueness of equilibrium. These properties are robust in the sense that they are true not just of our three-player example, but more generally. First, however, we draw attention to features, not shown in our particular example, that may characterize the graph.

First, we should note that, if all players were identical, then at the point $(\overline{p}, E)$, all would switch from pure appropriation to production-appropriation. In this case, the graph immediately becomes vertical, since the set A is empty and the consistency condition takes the form $\sum_{i \in I} \frac{p^A - c_i}{p^A + a_i} = 1$. This uniquely determines the value $\underline{p}$. In this situation, there is no strictly upward-sloping segment of the graph. It takes the form shown in Panel (a) of Figure 4, and $\underline{p} = \overline{p}$.

Second, it is possible that, as we reduce p along a strictly upward-sloping segment of the graph, an allocation is reached at which all existing producer-appropriators switch to pure production, leaving a non-empty set of pure appropriators. This happens in Panel (b) of Figure 4, and produces a horizontal step in the graph at a value of G that is strictly less than E.

Neither of these possibilities destroy the continuity and slope properties that will be used in our analysis of existence and uniqueness. Our conclusions concerning the set of appropriatively consistent allocations may be summarized as follows. The appropriative consistency condition implies a mapping from $p \in [\underline{p}, \infty)$ to $G \in [0, E]$, which we denote by $\Gamma^A(p)$, that has a closed graph and is everywhere non-decreasing.
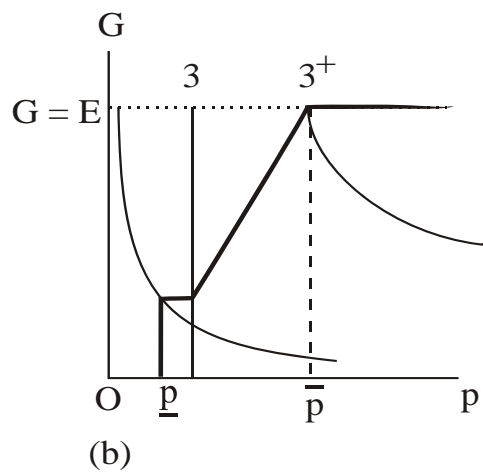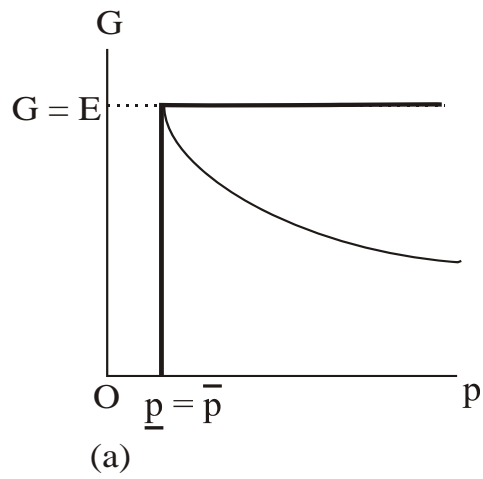
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Figure 4:

4.2 Productive Consistency

The graph implied by the productive consistency requirement (11) can be traced out simply. Note three features of this condition. First, even if either PA or A is empty, so that one or other of the summation terms disappears from the left hand side, the condition still contains both p^P and G^P. Second, if G alone increases, the right hand side will fall by at least as much as the left hand side, since the relevant summation term can be rewritten as $\sum_{i \in I} \frac{a_i e_i}{G^P} = \sum_{i \in A} \frac{a_i e_i}{G^P} + \sum_{i \notin A} \frac{a_i e_i}{G^P}$. Finally, an increase in p alone leads to a fall in the right hand side, and an increase in the left hand side, of (11). These observations imply that the graph of this condition is everywhere strictly downward-sloping.

Our conclusions concerning the set of productively consistent allocations may be summarized as follows. The productive consistency condition implies a single-valued mapping from $p \in \left[\underline{\underline{p}}, \infty\right)$ to $G \in [0, E]$, where $\underline{\underline{p}} < \overline{p}$. This mapping, denoted by $\Gamma^P(p)$, has a closed graph and is everywhere strictly decreasing.

We now state our first proposition:

Proposition 1 *The n-player production-appropriation model possesses a unique Nash equilibrium in pure strategies.*

Proof. *The groundwork has all been done. We have established that (8) implies a mapping $\Gamma^A(p)$ and that (10) implies a mapping $\Gamma^P(p)$. Their properties ensure the existence of a unique value of p for which $\Gamma^A(p) = \Gamma^P(p)$.* ∎

The thick continuous lines in Figure 5 are the graphs of $\Gamma^A(p)$ and $\Gamma^P(p)$ for given values of productivities and endowments. Their continuity and slope properties ensure a unique intersection, that determines the equilibrium values of p and G.

5 The Structure of Equilibrium

Before undertaking comparative static analysis of the unique equilibrium in the model, we should draw attention to a number of features of a given equilibrium. Our formulation, by using the same aggregates as arguments in every player's payoff and behavioral functions, makes it easy to read off how differences in parameters across players map into differences in their behavior and payoffs at equilibrium. Recall that player i's payoff and behavior can be described respectively by

$$v_i(g_i, p, G) = \lambda a_i(e_i - \widehat{g}_i) + (1 - \lambda) p \widehat{g}_i - b_i \widehat{g}_i \tag{12}$$

where $\widehat{g}_i = \rho_i(p, G)$ as in (7).

5.1 Endowment Differences

Consider a given equilibrium allocation, (p^*, G^*). We are interested in how player i's endowment influences her equilibrium payoff. We are interested in comparing the equilibrium payoffs across a subset of players who have identical productivity and theft aversion parameters, but who enjoy different endowments. We will denote the common parameter values by
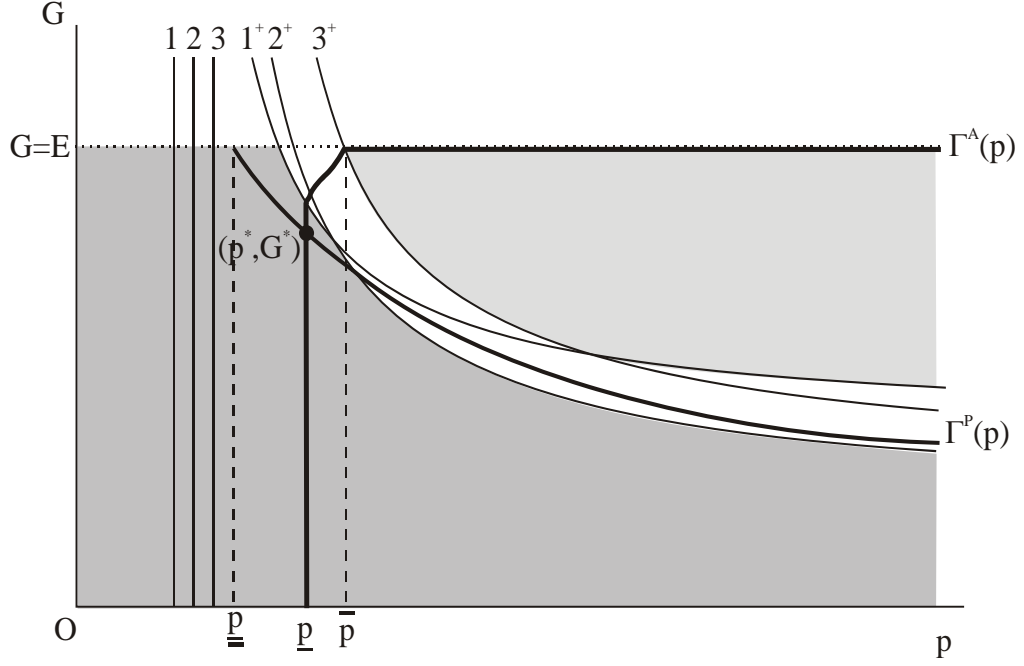
Figure 5:

the unindexed quantities a and b. Two possibilities arise. For players within the subset, it is possible that $p^* \leq c = \frac{\lambda a - b}{1 - \lambda}$. In this case, whatever player i's endowment, she will choose $\widehat{g}_i = 0$, and her realized payoff is

$$v_i^* = \lambda a e_i.$$

Now suppose, instead, that $p^* > c$. In this case, the value of i's endowment determines whether she is a producer-appropriator or a pure appropriator at equilibrium:

$$
\begin{aligned}
e_i \geq \left(\tfrac{p^*-c}{p^*+a}\right) G^* \quad &\Longrightarrow \widehat{g}_i = \left(\tfrac{p^*-c}{p^*+a}\right) G^* \\
&\Longrightarrow v_i^* = \lambda a \left(e_i - \left(\tfrac{p^*-c}{p^*+a}\right) G^*\right) + \left(\tfrac{p^*-c}{p^*+a}\right)(1-\lambda) p^* G^*, \\
e_i < \left(\tfrac{p^*-c}{p^*+a}\right) G^* \quad &\Longrightarrow \widehat{g}_i = e_i \\
&\Longrightarrow v_i^* = [(1-\lambda) p^* - b] e_i.
\end{aligned}
$$

Our analysis enables us to infer that, if $\lambda > 0$ and players j and k are identical except that $e_j > e_k$, then at equilibrium player j enjoys a higher payoff than player k. This is not surprising. However, we can make a stronger statement, which is less obvious and more interesting.

Proposition 2 *Suppose $\lambda > 0$, and players j and k are identical except that $e_j > e_k$.*

(i) If both players are pure appropriators, then the ratio of their equilibrium payoffs v_j^*/v_k^* equals the ratio of their endowments e_j/e_k.

(ii) If player j is a producer-appropriator, then the ratio of equilibrium payoffs is less than that of endowments.

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Part (ii) of this proposition captures the thrust of Hirshleifer's 'Weak' Paradox of Power—see Hirshleifer ([13], p. 52)[4]. However, the ease with which we can accommodate many heterogeneous players allows us to extend his observation to the more general many-player setting. Some of the benefits from j's greater endowment flow to other players. To this extent, the presence of appropriation has an equalizing effect on the equilibrium outcome.

Now consider the situation in which $\lambda = 0$, so that all of a player's output immediately goes into the common pool. Though somewhat extreme, this is a feature of a number of existing models of appropriative behavior. In this extreme case, a stronger proposition can be made:

Proposition 3 *Suppose $\lambda = 0$, and players j and k are identical except that $e_j > e_k$.*
(i) If both players are pure appropriators, then the ratio of their equilibrium payoffs equals the ratio of their endowments.
(ii) If both players are producer-appropriators, then $v_j^* = v_k^*$.

Part (ii) of this proposition captures the thrust of Hirshleifer's 'Strong' Paradox of Power, again placing it in a somewhat more general context. This property is very similar to one that characterizes the model of voluntary contributions to a public good. In that model, players who are making positive contributions to the public good in equilibrium, and whose preferences are identical, will enjoy the same real consumption bundle and hence the same utility level, even though their initial endowments differ.

Figure 6 summarizes the situations described in this subsection.

5.2   Productivity Differences

We now consider the implications of productivity differences across a subset of players whose endowments and attitudes towards thievery are identical. Again, we denote these common values by the unindexed quantities e and b. However, their productivities may differ.

For the moment, consider an extreme situation in which $\lambda = 0$. Then

$$v_i \left(\widehat{g}_i, p, G\right) = \widehat{g}_i p - b \widehat{g}_i$$

First, it is possible that, for all players in our subset, $p^* \leq b$. In this case, regardless of the value of i's productivity, $\widehat{g}_i = 0$ and $v_i^* = 0$.

Now suppose that $p^* > b$. Whether player i is a producer-appropriator or a pure appropriator at equilibrium depends on the value of a_i:

$$a_i < \frac{(p^*+b)G^*}{e} - p^* \quad \Longrightarrow \widehat{g}_i = e$$
$$\Longrightarrow v_i \left(\widehat{g}_i, p^*, G^*\right) = (p^* + b)\, e,$$
$$a_i \geq \frac{(p^*+b)G^*}{e} - p^* \quad \Longrightarrow \widehat{g}_i = \left(\frac{p^*+b}{p^*+a_i}\right) G^*$$
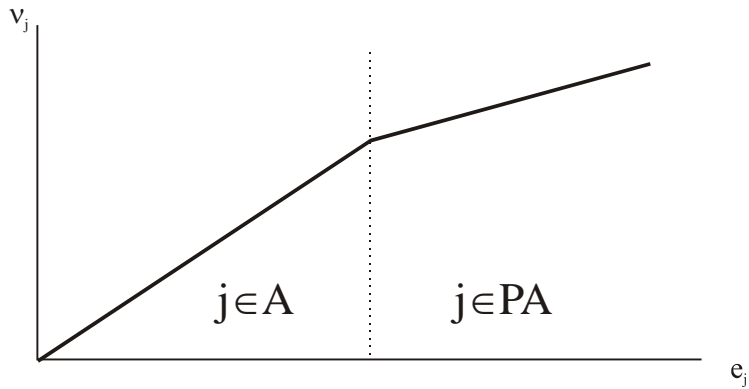$$\Longrightarrow v_i \left(\widehat{g}_i, p^*, G^*\right) = \frac{p^{*2}-b^2}{(p^*+a_i)} G^*.$$

Thus,

_____

[4]There is an unfortunate typographical error in this reference. The second of the two forms of the paradox that Hirshleifer presents should, presumably, be labeled the "weak" form.
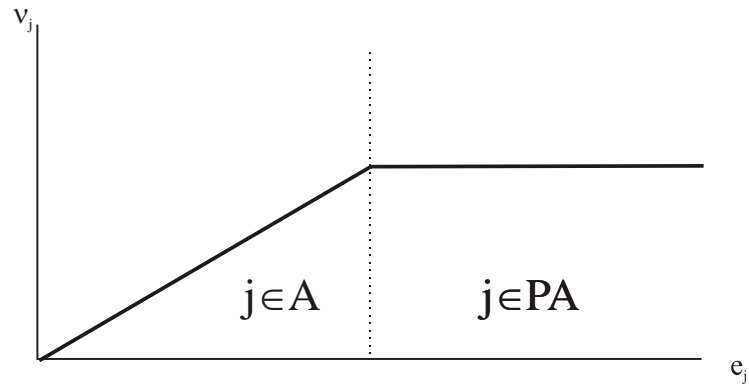
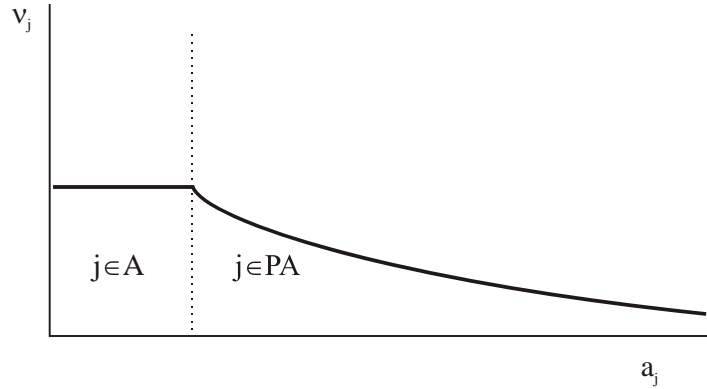Equilibrium payoff graphed against individual endowment:
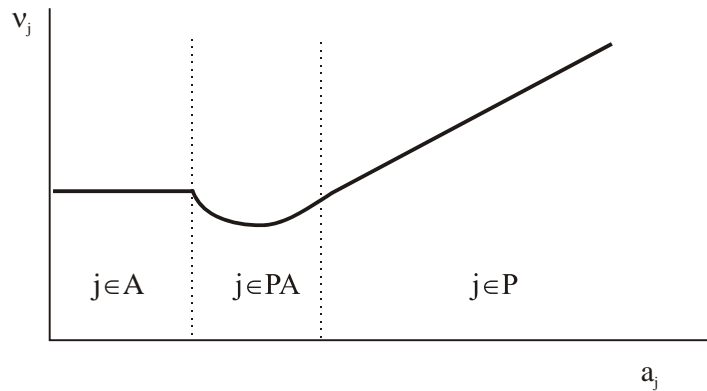$\lambda = 0$, $b = 0$, $e = 1$, $(p^*, G^*) = (1,10)$

Equilibrium payoff graphed against individual endowment:
$\lambda = \frac{1}{4}$, $b = 0$, $e = 1$, $(p^*, G^*) = (1,10)$

Figure 6:

Equilibrium payoff graphed against individual productivity:
$\lambda = 0$, $b = 0$, $e = 1$, $(p^*, G^*) = (1, 10)$

Equilibrium payoff graphed against individual productivity:
$\lambda = \frac{1}{4}$, $b = 0$, $e = 1$, $(p^*, G^*) = (1, 10)$

Figure 7:

Proposition 4 *If players j and k are identical except that $a_j > a_k$, and if player k is a producer-appropriator at equilibrium, then player j enjoys a lower equilibrium payoff than player k.*

In this model, producer-appropriators with higher productivities are strictly worse off in equilibrium than their fellows. This conclusion may seem surprising, but it parallels similar properties in the simple pure public good provision model. There, a player with higher productivity will make a higher contribution, and will be worse off than others, who are able to enjoy a free ride on her contribution. See Cornes and Hartley [2]. Here, a producer-appropriator with higher productivity will devote fewer resources to appropriation. She will produce more output which, as in the public good model, generates an positive externality for others. Hence, others can enjoy a free ride, having a larger pool from which to appropriate for their consumption.

If $\lambda > 0$, so that each player can enjoy some of her own production without having to

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engage in appropriation, this free-riding possibility is mitigated, but not eliminated. The second panel of Figure 7 graphs the equilibrium payoff against individual productivity values for $\lambda > 0$. Within the interval that is consistent with a player being a producer-appropriator, there is a subinterval over which, the higher the productivity, the lower the equilibrium payoff. But, as a_i rises further, the expected positive relationship between individual productivity and equilibrium payoff is restored.

6 Comparative Static Exercises

Having established the existence of a unique equilibrium, we can now do comparative static exercises. In each case, our approach will be to examine the shifts in the graphs of the two equilibrium requirements that are implied by the exogenous shock, and hence in their sole intersection. Before putting the approach to work in the presence of heterogeneous players, we will first work through the simple case in which all players are identical, and the equilibrium is symmetric.

6.1 n Identical Players

Suppose all players are identical, so that $(a_i, b_i, c_i, e_i) = (a, b, c, e)$. A Nash equilibrium allocation (p^*, G^*) must satisfy the two consistency conditions:

$$\text{AC:} \quad n\left(\frac{p^* - c}{p^* + a}\right) = 1 \tag{13}$$

and

$$\text{PC:} \quad na\left(\frac{p^* - c}{p^* + a}\right) = \frac{nae}{G^*} - p^*. \tag{14}$$

We can immediately infer that the equilibrium values are

$$p^* = \frac{a + cn}{n - 1} \tag{15}$$

and

$$G^* = \frac{ane}{p - an\frac{c-p}{a+p}} = (n - 1)\frac{ae}{a + c}. \tag{16}$$

For completeness, we also solve for the equilibrium values of total output and per capita utility:

$$Y^* = p^* G^* = ae\frac{a + nc}{a + c} \tag{17}$$

and

$$u^* = \lambda ae + (1 - \lambda)\frac{ae}{n}. \tag{18}$$

Recall that the parameter c is a composite parameter so that a change in, say, the common productivity parameter a influences the equilibrium value of p through two routes: directly, and indirectly through its effect on c.
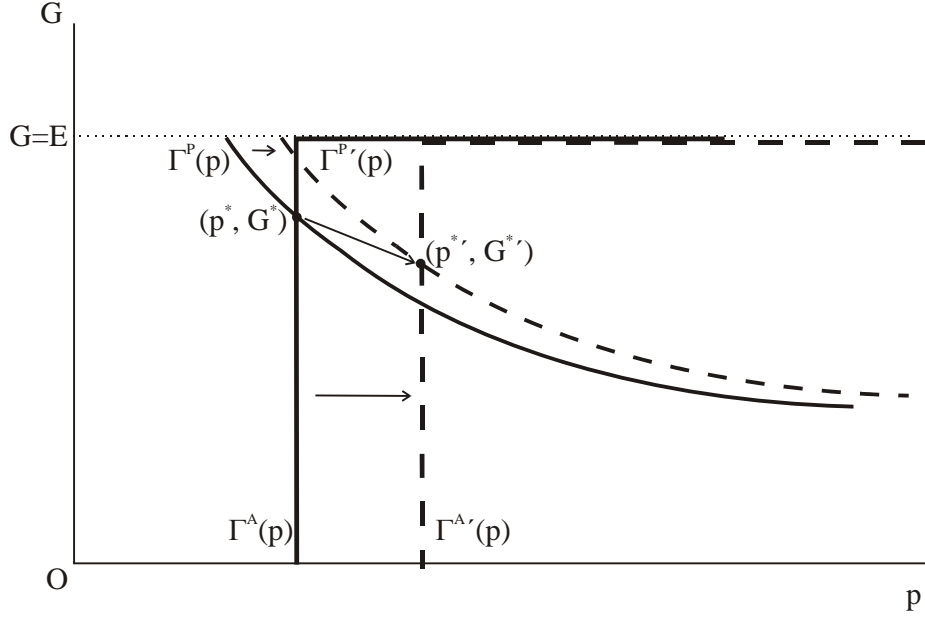
Figure 8:

Comparative static results can be read off directly from these expressions. However, let us first look at them in terms of the graphs of $\Gamma^A(p)$ and $\Gamma^P(p)$. Initially, these are as shown in Figure 8, and the equilibrium is at (p^*, G^*). Now suppose λ, the proportion of a player's output that does not go into the common pool, rises.

This implies an increase in the parameter c, which must be accompanied by an increase in p if the appropriative consistency condition is to be maintained. The graph of $\Gamma^A(p)$ shifts to the right. So too must the graph of $\Gamma^P(p)$, but by less. This is most clearly seen by dividing both sides of (14) by na. The change in c has the same effect on the LHS of (14) as it does on the LHS of (13). So, too, does a given response on the part of p. But an increase in p also reduces the RHS of (14). Hence, the increase in p required to maintain productive consistency at an unchanged value of G is smaller than that required to maintain appropriative consistency. Consequently, the equilibrium value of p must now be higher, and that of G lower, than initially. Returning to the equilibrium equations, observe from (17), too, that the equilibrium value of total output must now be higher than initially. Finally, the equilibrium utility level has also risen.

The effects of exogenous changes in the other parameters on equilibrium when players are identical may be analyzed, and graphically depicted, in the same way. One point worth noting is the implication of raising b, the aversion to thieving. A higher value of b implies a higher value of c. It thereby implies higher values for p and Y, and a lower value for G. But the representative player's utility in equilibrium is unchanged. The precise way in which the shift of preferences undoes the change in real resource allocation reflects our particular formulation of aversion to thieving within the players' payoff functions.

Before moving on to examine the implications of heterogeneity, we present a tabular summary of the comparative static properties of the model when players are identical:

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	p	G	Y	u
a	$+$	$+$	$+$	$+$
b	$+$	$-$	$+$	0
e	$+$	$+$	$+$	$+$
λ	$+$	$-$	$+$	$+$
n	$-$	$+$	$+$	$-$

6.2 n Heterogeneous Players

Heterogeneity has several notable implications. First, it introduces the possibility that, at a given equilibrium, different players are in different regimes—pure appropriators, producer-appropriators and pure producers may all coexist in equilibrium. Second, even within the set, say, of producer-appropriators, differences in the parameters across those players may lead to different behavioral and welfare responses to a given exogenous shock. Finally, external shocks may well change the assignment of players to the sets (A, PA, P).

The various comparative static analyses that follow are intended to be illustrative rather than exhaustive. We concentrate on drawing attention to properties that survive the transition from identical to heterogeneous player games, and to those novel features that emerge in the presence of heterogeneity. For simplicity, unless we explicitly state otherwise, we will generally consider situations in which the shock does not switch any individual player from one regime to another.

In the examples that follow, inferences about the shifts in the graphs of the consistency requirements all flow in a simple way from inspection of the conditions (9) and (11).

6.2.1 Changes in λ

A change in λ affects both of the consistency requirements through its effect on the parameters c_i for all $i \in PA$. A higher value of λ, other things held equal, implies a higher value of c_i for all i. In both equations, the increase in λ must be accompanied by an increase in p for any given value of G in order to maintain consistency. Therefore, the equilibrium value of p must now be higher. We have already seen that, if all players are identical, the horizontal rightward shift in Γ^A is greater than the shift in Γ^P, and we will suppose that this presumption remains the case here[5]. In this case, the equilibrium value of p will be higher, and that of G lower, than initially.

The implications of an increase in λ for individual equilibrium payoff values are not entirely straightforward. Suppose that, at the initial equilibrium, player i is a pure appropriator. Then, initially, $\widehat{g}_i = e_i$ and

$$v_i (g_i, p, G) = [(1 - \lambda) p - b_i] e_i$$

For simplicity, consider a small change in λ, and suppose that she remains a pure appropriator after the change. Then the change in her payoff is

$$dv_i = -p d\lambda + (1 - \lambda) dp$$

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[5]This may not be true in the presence of many heterogeneous players.

The first term in this expression captures that fact that the increase in λ reduces the share of output that goes into the common pool, thereby hurting a pure appropriator. But the endogenous response to the increase in λ involves an increase in p—and, indeed, Y—which benefits a pure appropriator. The sign of the net effect is unclear.

The change in a pure producer's payoff is straightforward. Such a player's payoff is

$$v_i\left(g_i, p, G\right) = \lambda a_i e_i$$

An increase in λ unambiguously benefits such a player, who is able to retain a higher proportion of her own output.

Considerations of continuity suggest that producer-appropriators may be either hurt or benefitted by an increase in λ.

6.2.2 Changes in individual resource endowments

Suppose a single player enjoys an exogenous endowment increase. The implications of this for the equilibrium allocation depend in the first place on which regime this player is in initially.

If, initially, she is a pure producer then the appropriative consistency condition is not affected by the shock, and its graph remains fixed. However, for any given value of p, the increase in the player's endowment raises the right hand side of the productive consistency condition, and must be accompanied by an increase in G in order to maintain productive consistency. The equilibrium values of both p and G will rise.

At the other extreme, suppose that the player is initially a pure appropriator. Then, by itself, the endowment increase raises the left hand side of the appropriative consistency requirement. Maintenance of appropriative consistency then requires that, at any given value of p, G must now be higher. So the graph Γ^A shifts upwards. However, the productive consistency requirement is undisturbed by the endowment change, since, at unchanged values of p and G, the left-hand side changes by the same magnitude as the right-hand-side. In this case, therefore, the equilibrium value of p falls and that of G rises.

Finally, suppose the player experiencing the endowment shock is a producer-appropriator. The appropriative consistency requirement is unaffected by the shock, but the productive consistency requirement is affected. By itself, the increase in endowment raises the RHS. To maintain productive consistency, G must be higher at any given value of p. In short, the graph Γ^P shifts upwards.

6.2.3 Changes in individual productivities

First, suppose that a_i rises and that $i \in P$. The implications are similar to those of an endowment shock. Γ^A is unaffected by the change in a_i. However, it increases the value of the RHS of the productive consistency requirement. At any given value of p, G must rise to maintain productive consistency. So Γ^P shifts upwards. The equilibrium values of both p and G rise.

Now suppose, instead, that $i \in A$. Again, Γ^A is unaffected by this shock. So, too, is Γ^P, since the change in a_i changes both the LHS and the RHS of (11) by the same value at

unchanged values of p and G. The productivity of a pure appropriator is irrelevant in this model.

Finally, suppose that $i \in PA$. The LHS of (9) is reduced by this shock. For any given value of G, p must rise to maintain appropriative consistency. Γ^A therefore shifts to the right. The effect on the productive consistency requirement is a little more subtle. An increase in a_i, $i \in PA$, affects both sides of (11). We will argue that it increases the RHS by more than the LHS. Its effect on the LHS works through the term $a_i \frac{p-c_i}{p+a_i}$. This may be decomposed as follows:

$$\frac{d}{da_i}\left(a_i\frac{p-c_i}{p+a_i}\right) = \frac{p-c_i}{p+a_i} + a_i\frac{d}{da_i}\left(\frac{p-c_i}{p+a_i}\right).$$

The fact that $i \in PA$ necessarily implies that $\frac{p-c_i}{p+a_i} \leq \frac{e_i}{G}$. Thus

$$\frac{d}{da_i}\left(a_i\frac{p-c_i}{p+a_i}\right) \leq \frac{e_i}{G} + a_i\frac{d}{da_i}\left(\frac{p-c_i}{p+a_i}\right).$$

Recall that the parameter c_i includes the productivity parameter a_i. It is easily confirmed that $\frac{d}{da_i}\left(\frac{p-c_i}{p+a_i}\right) < 0$. We conclude that

$$\frac{d}{da_i}\left(a_i\frac{p-c_i}{p+a_i}\right) < \frac{e_i}{G}.$$

Thus, the productivity shock alone implies that the LHS of (11) becomes strictly less than the RHS. To restore productive consistency, p must now be lower at any given value of G. In short, Γ^P shifts to the left.

6.2.4 Changes in b_i

An interesting possibility that arises when there are heterogeneous players is that an increase in the aversion to thieving on the part of some proper subset of players, with an unchanged level of aversion on the part of all others, may lead to an increase in the aggregate level of appropriation.

This is best seen by looking at a numerical example. Suppose that there are two types of player, 1 and 2. The initial values of the parameters of the model are

$$\begin{aligned}
\lambda &= 0 \\
(b_1, b_2) &= \left(\frac{11}{10}, 1\right) \\
(a_1, a_2) &= (1, 5) \\
(n_1, n_2) &= (8, 2) \\
n_1a_1e_1 + n_2a_2e_2 &= 30
\end{aligned}$$

Suppose, too, that individual endowments, which we have not specified precisely, are distributed in a way that is consistent with both types being producer-appropriators at the initial equilibrium. Then the equations of equilibrium are

$$[\text{AC}]: 8\left(\frac{p^* - \frac{11}{10}}{p^* + 1}\right) + 2\left(\frac{p^* - 1}{p^* + 5}\right) = 1$$

and

$$[\text{PC}]: 8 \left(\frac{p^* - \frac{11}{10}}{p^* + 1} \right) + 10 \left(\frac{p^* - 1}{p^* + 5} \right) = \frac{30}{G^*} - p^*.$$

The unique solution is $(p, G) = (1.36, 10.65)$. Now, as a consequence of an exogenous moral crusade, or policy intervention, b_2 rises to become equal to that of type 1.

If both types remain producer-appropriators, the new solution is $(p, G) = (1.37, 11.06)$. Both aggregate output and the aggregate level of stealing have risen. This is possible because the more productive type has substituted out of appropriation into production.

6.2.5 Endowment Redistribution

Return for a moment to the economy with n identical players, and consider the implied symmetric equilibrium, at which every player has an endowment of $\overline{e}$ and is a producer-appropriator. Now consider redistributions of initial endowments. For given values of her other parameters, an increase in a player's endowment has no effect on her appropriation choice. A reduction in her endowment has no effect on that choice unless it reduces her endowment to the level at which she flips over to being a pure appropriator. Thus, any income distribution that satisfies the inequality

$$e_i \geq \frac{(n-1)}{n} \frac{a\overline{e}}{a+c} \text{ for all } i \in I \tag{19}$$

leaves every player's choice of appropriation, and therefore the aggregate level of appropriation, unchanged. This in turn implies that each player's production changes in proportion to their endowment change. Under our assumptions, therefore, aggregate output is unchanged.

Starting at the symmetric equilibrium with n identical players, then, aggregate variables are unaffected by an endowment redistribution that respects inequality (19). If, in addition, $\lambda = 0$, then the equilibrium allocation is unaffected by the initial endowment redistribution. However, if $\lambda > 0$, such redistribution will have an effect on individual equilibrium payoffs, even though the aggregate values of Y and G remain unchanged. Recall equation (2). This shows that the change in endowment has a direct effect on a player's equilibrium payoff, which works through the first term on the RHS. For example, an increase in player i's endowment is devoted entirely to an increase in her production of output, and the fraction λ of this change is enjoyed by her as additional consumption.

Proposition 5 ('Full neutrality') *If all players are identical except for their endowments, and if $\lambda = 0$, then any distribution of endowment satisfying*

$$e_i \geq \frac{(n-1)}{n} \frac{a}{a+b} \overline{e}$$

for all $i \in I$ leads to the same real equilibrium, in which every player enjoys the same payoff.

A direct implication of this proposition is that, as $n \to \infty$ and if $b \to 0$, then the scope for such neutral redistributions becomes smaller.

If the possibility of a strictly positive value of λ is reintroduced, then we have a 'semi-neutrality' result:

Proposition 6 ('Semi-neutrality') *If all players are identical except for their endowments, and if $\lambda > 0$, then any distribution of endowment satisfying*

$$e_i \geq \frac{(n-1)}{n} \frac{a}{a+c} \overline{e}$$

for all $i \in I$ leads to the same equilibrium of the aggregates p, G and Y. Player i's equilibrium payoff is positively related to her endowment.

Redistributions that violate the above inequality will have real effects.

7 Concluding Remarks

The purpose of this paper has been primarily to present a promising method of analyzing games of production and appropriation—a method that can cope with three or more heterogeneous players. We have established existence and uniqueness of Nash equilibrium in such a model, and have presented some simple comparative static exercises in order to show that this method does, indeed, provide a promising alternative way of organizing our thoughts about models of conflict involving many heterogeneous players. In addition to exploring further comparative static and normative issues, we intend to explore extensions of the model in various directions. Accommodating diminishing returns in the production of output represents an important extension, as does a more general formulation of individual preferences.

We emphasize again that, in addition to the particular application of our approach to the modeling of conflictual situations, there are many other applications that lend themselves naturally to our modeling approach through the use of two aggregates. Dickson and Hartley [5], [6] have applied this approach to models of market games and bilateral oligopoly that share the feature of being fully specified by two aggregates. Other applications remain to be explored. For example, suppose a group of players, each with her own idiosyncratic innate productivity, are involved in a joint productive enterprise. Then the aggregate output which will be shared amongst them will depend upon the sum of their effective inputs. In addition, if the rule for sharing that output depends in part on the observed number of hours that each provides, then the total number of hours—which will be a weighted sum of their effective input levels—will also be argument of individual payoff and behavioral functions. A further application is to an oligopolistic industry in which production also generates pollution as a by-product, or else depletes a scarce open access resource. If firms differ with respect to their roles as polluters or depleters, then again a two-aggregate structure naturally emerges. Many further potential applications to models involving many heterogeneous players spring readily to mind, in which the present approach avoids the extreme limitations imposed by the proliferation of dimensions when the best response formulation is used.

In short, the current paper is no more than a preliminary step in exploring the usefulness of studying games with more than a single aggregate, and an invitation to others to explore our method further. However, it is a promising step.

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