Two-aggregate games: Demonstration using a production-appropriation model

Richard Cornes
Roger Hartley
Yuji Tamura

DISCUSSION PAPER NO. 696
JANUARY 2017
Two-aggregate games: Demonstration using a production-appropriation model

Richard Cornes† Roger Hartley‡ Yuji Tamura§

January 14, 2017

Abstract

We expand the scope of the two-aggregate method by applying it to a situation where many heterogeneous players are free to contribute to both aggregates. Such situations naturally arise in various resource allocation problems. Hence our method is useful in many applications. A production-appropriation model is employed to illustrate how the problem of establishing the Nash equilibrium can be reduced from solving $n > 2$ best response functions in $n$ unknowns to solving two consistency conditions in two unknowns. We then conduct a comparative static exercise that the conventional approach could not handle easily—if at all—to demonstrate the power of our method.

JEL classification codes: C72, D74

Keywords: noncooperative game, aggregate game, conflict, appropriation

---

*We are grateful to Tim Hatton, Alex Dickson, Yvette Kirby, participants of seminars and workshops at ANU, Manchester, Massey, and two anonymous referees for useful comments. Cornes’ research was supported by the F. H. Gruen endowment. Remaining errors are ours. [Forthcoming in the *Scandinavian Journal of Economics*]

†Research School of Economics, Australian National University
‡Department of Economics, School of Social Sciences, University of Manchester
§Department of Economics and Finance, La Trobe Business School, La Trobe University (corresponding author: y.tamura@latrobe.edu.au)
1 Introduction

The analysis of several economic problems has benefitted from exploiting the observation that, in each case, a single aggregate provides a sufficient statistic for solving the implied noncooperative game with many heterogeneous players. Dickson and Hartley (2008; 2013) made an important contribution to this aggregate-game literature through the analysis of market games where a single aggregate is not sufficient to describe each player’s optimal behavior, but just two aggregates can describe it. In their models, each player is allowed to contribute to only one of the two aggregates, which is not suitable for examining many situations of interest. We show that the two-aggregate method can readily accommodate situations where each player is free to contribute to both aggregates, thereby expanding the scope of the method and opening up new analytical possibilities for many applications. A model of production and appropriation is used to illustrate the usefulness of the method.

Many economic models involve noncooperative games with an aggregate structure that can be exploited to facilitate their analysis. By conditioning every player’s behavior on a common aggregate, instead of the sum of choices of all others, it is possible to avoid the proliferation of dimensions associated with the use of best response functions. As a result, the presence of many heterogeneous players does not necessarily complicate the analysis. Early use of the single-aggregate method appears in Szidarovszky and Yakowitz (1977) and Novshek (1985) in the context of proving the existence of Cournot equilibrium. Corchón (2001) points out this structure in a wide range of applications. Cornes and Hartley have exploited the method in analyzing contests (2003; 2005; 2012) and public goods (2007a; 2007b).¹

The scope of this analytically attractive approach was considerably broadened by Dickson and Hartley (2008; 2013) who examined the Nash equilibrium of market games that not one but just two aggregates can describe, regardless of how many heterogeneous players there are. In their models, there are two types of goods. Each player is endowed with only one of these two and decides how many units of the endowment to trade

for the other good. Each market participant can thus contribute to only one of the two aggregate supplies, and the initial endowment allocation exogenously determines the group partition.

We expand the scope of the aggregate-game approach further by showing how to apply the two-aggregate method to a situation where every one of many heterogeneous players is allowed to contribute to both aggregates. In our model, unlike in Dickson and Hartley’s (2008; 2013), the initially allocated endowment does not prevent each player from contributing to both aggregates. Players are free to contribute to either one of the two, or both, according to their payoff maximization. This extension is significant because many situations of interest indeed involve a resource allocation to multiple activities. Furthermore, this feature enables us to sort players into groups endogenously according to their resource allocation decisions. Our model for illustration is of production and appropriation, which is chosen because of the feature shared by several models in the literature that each player can use her initial endowment, such as labor time, to both produce a consumption good and seize the output of the other players.

This production-appropriation application is of interest also in its own right. Much attention has been devoted to developing and exploring formal models in which individuals or groups may find it worthwhile to devote part, or even all, of their resources to the appropriation of the hard-won output of others rather than to the production of net output. Edgeworth (1881) referred to the use of conflict, or war, as a significant alternative resource allocation device. Pareto, too, noted the significance of appropriation as a means of acquiring economic goods.2 Haavelmo (1954), Bush and Mayer (1974), Skogh and Stewart (1982) and Usher (1987; 1989) are early contributors of formal models of appropriation. Hirshleifer (1991; 1995), Grossman (1991; 1994; 2001), Grossman and Kim (1995), Skaperdas (1992) and Garfinkel and Skaperdas (2004; 2007) followed.3

As the survey by Anderton and Carter (2009) shows, the literature on the economics of conflict is now large and diverse. Yet, much of the literature restricts attention to

---

2See the quotes in Usher (1987) and Hirshleifer (2001: 1).
3Hirshleifer (2001) is a collection of his work in this literature. See also Garfinkel and Skaperdas (1996).
two-player models. Alternatively, researchers either assume many identical players and examine a symmetric equilibrium, or resort to numerical simulation. This is understandable, as many models of conflict are intricate. Nevertheless, all three approaches raise concerns about the robustness of conclusions reached, and the scope of the analysis in terms of the kinds of questions that it can address. To our best knowledge, there does not seem to exist an analytical demonstration of the existence of a unique Nash equilibrium in pure strategies in the presence of many heterogeneous players. Hence a successful application of the aggregate-game method can benefit the literature significantly.

We present an alternative way of writing down a class of production-appropriation models by building on the two-player model of Garfinkel and Skaperdas (2007). We incorporate the possibility that each player automatically has access to a certain proportion of her own output as a consumer without having to contest that portion. This feature is often missing in the literature but can help us understand cases of imperfect anarchy. We show that Hirshleifer’s (1991) weak paradox of power is robust in the presence of many heterogeneous players under complete anarchy, but not under partial anarchy. Garfinkel and Skaperdas (2007) points out a similar paradox with respect to not endowment difference but productivity difference. We show that this paradox also does not necessarily hold when there are many heterogeneous players and a fraction of each player’s own output is secure.

First, we describe each player’s production and appropriation decisions by the total output and the total appropriation activity in the game. We then exploit the two-aggregate method to establish the existence of a unique Nash equilibrium in pure strategies. Finally, we present one illustrative comparative static exercise that demonstrates the advantage of the aggregate-game method. In this exercise, we examine the consequences of letting an additional agent play the game. Some studies in the existing literature have looked at the effects of increasing the number of identical players on the symmetric equilibrium. Our approach does not limit us to the case of identical players, and enables us to answer questions that could not be handled previously.\textsuperscript{4} We add different entrants,

\textsuperscript{4}Our model is a simple one and does not cover some features of more elaborate models in the literature, as the main aim of this paper is not a major contribution to the economics of conflict but a demonstration
one at a time, to a pool of heterogeneous players and examine how the entry affects
the equilibrium behaviors and payoffs of the existing players. The analysis suggests that
some, but not all, existing players may benefit from the participation of one more player,
even though this new player decides to devote all her resources to seize their output.

2 Model

There are \( n > 2 \) players, and the set of all players is \( I \). Player \( i \)'s exogenous endowment,
\( e_i \), is divided between two activities: production and appropriation. We denote the level
of \( i \)'s production activity by \( x_i \). Player \( i \)'s allocation to appropriation activity is \( g_i \). Player
\( i \)'s constraint set requires that \( e_i = x_i + g_i \). We denote the aggregate level of appropriation
activity by \( G = \sum_{j=1}^{n} g_j \).

We denote the output of player \( i \) by \( y_i = a_i x_i \), where \( a_i \) is her exogenous productivity
parameter,\(^5\) and \( Y = \sum_{j=1}^{n} a_j x_j \). An exogenously given proportion, \( \lambda \in [0, 1) \), of player
\( i \)'s output is available for consumption by that player. The remainder goes into a common
pool, and players may devote resources to expropriation from this pool. The share of the
common pool that is enjoyed as consumption by player \( i \) reflects her share of the total
effort devoted to appropriation, \( g_i / G \). Her total level of consumption is given by

\[
z_i = \lambda y_i + \frac{g_i}{G} (1 - \lambda) Y. \tag{1}
\]

Garfinkel and Skaperdas (2007) interpret \( g_i / G \) as the probability of \( i \)'s winning the output
in the common pool, rather than the proportion of that output that is acquired with
certainty. Either interpretation is admissible given risk neutrality on the part of all
players. We assume that player \( i \)'s utility is a linear function of her consumption and the
effort that she devotes to appropriation:

\[
u_i (z_i, g_i) = z_i - b_i g_i
\]

\(^5\)The productivity heterogeneity generates a need for two aggregates in this model. If the parameter
is common to all players, we have \( Y = (E - G) a \) and will need only a single aggregate.
where the parameter $b_i$ captures the cost to player $i$ per unit of appropriation. A player is fully described by the trio $(a_i, b_i, e_i)$. Using (1) to substitute for $z_i$, player $i$’s payoff can be written as

$$v_i(g_i, G; Y) = a_i(e_i - g_i) + \frac{g_i}{G} (1 - \lambda) Y - b_i g_i.$$  \hfill (2)

We follow the existing literature in exploring the properties of pure-strategy Nash equilibrium. Equation (2) expresses player $i$’s payoff as a function of her own action and two aggregate quantities. The conventional way of expressing $i$’s payoff in the noncooperative game context would be

$$\pi_i(g_i, G-i, Y-i) = \lambda a_i(e_i - g_i) + \frac{g_i}{G-i + g_i} (1 - \lambda) (Y-i + a_i(e_i - g_i)) - b_i g_i$$

where $G-i \equiv G - g_i$ and $Y-i \equiv Y - y_i$. The quantities $G-i$ and $Y-i$ are parametrically fixed when describing player $i$’s behavior in the noncooperative game, and they would appear as arguments in that player’s best response function.

An alternative interpretation of the payoff function (2) is suggested by the observation that, if total output $Y$ were exogenously given and $\lambda$ were zero, $i$’s payoff function would have precisely the same form as that encountered in the theory of contests involving risk neutral players. Hence our formulation extends contest theory by allowing the magnitude of the prize to depend endogenously on choices made by the individual contestants. Each player chooses how much effort to devote to increasing the total size of the pie, and how much to devote to increasing her share of that pie.

One interpretation of $\lambda$ sees it as reflecting the extent to which the state provides security to individuals by ensuring their private property rights over some part of their own output. A situation in which $\lambda = 0$ reflects an anarchic world in which no individual’s output is safe from predation by others. A strictly positive value of $\lambda$ reflects the existence of institutions that protect a proportion of an individual’s output from predation by others.

\footnote{This interpretation implicitly assumes that exogenous protection is applied to all players equally. If protection is assumed to benefit different players differently, the parameter should be personalized.}
An alternative interpretation turns the role of the state on its head. Here, the parameter $\lambda$ may be interpreted as reflecting the presence of a confiscatory state, which taxes a fraction, $1 - \lambda$, of individual output. Players then devote resources to conflict, maybe in the form of lobbying, in order to acquire for their own consumption some portion of the confiscated output. Our model draws attention to an efficiency cost associated with taxation, in the form of the socially unproductive conflict that it generates through encouraging the contest among players to claw back the resources that the government has expropriated. This is in the spirit of rentseeking contests of the kind suggested by Tullock (1980), and is distinct from the standard deadweight loss, from which we abstract, associated with distortionary taxation.

The parameter $b_i$ may be interpreted as reflecting the degree to which prevailing norms engender a feeling of guilt, or attitude of opprobrium, on the part of player $i$ associated with the act of appropriating the output of others. An alternative interpretation is that, rather than being motivated from within, players are discouraged from stealing by policies, not explicitly modeled here, that influence the probability of detection and the severity of punishment in the event of detection. Whatever the interpretation, one can pose questions about whether policies or campaigns to increase the value of such feelings of guilt may make individuals in society better off, and indeed about the socially optimal value of such a parameter.

3 Player $i$’s best-response resource allocation

Player $i$ chooses $g_i \in [0, e_i]$ to maximize her payoff. The first-order condition characterizing an interior solution for player $i$ is

More generally, $\lambda_i$ is likely to be a function of the player’s defensive effort relative to other players’ grabbing efforts. In this paper, we abstract from this important endogeneity. Skogh and Stuart (1982), Grossman and Kim (1995) and Grossman (2001) contain defense as a decision variable in their models, but their analysis is limited to either two players or many identical players. Lasserre and Soubeyran (2003) present an attempt to endogenise $\lambda$. Unfortunately, their restrictive assumptions involved in endogenising $\lambda$ result in an unconvincing equilibrium where all players concentrate on appropriation even though there is nothing to appropriate in the economy.
\[
\frac{\partial \pi_i}{\partial g_i} = \frac{\partial v_i}{\partial g_i} + \frac{\partial v_i}{\partial G} \frac{\partial G}{\partial g_i} + \frac{\partial v_i}{\partial Y} \frac{\partial Y}{\partial g_i} = 0. \tag{3}
\]

The level of \(g_i\) that satisfies this first-order condition is denoted by

\[
\tilde{g}_i = \left[ (1 - \lambda) Y - (\lambda a_i + b_i) G \right] \frac{G}{(1 - \lambda) (Y + a_i G)},
\]

which can be written more economically as

\[
\tilde{g}_i = \frac{Y - c_i G}{Y + a_i G}, \tag{4}
\]

where

\[
c_i = \frac{\lambda a_i + b_i}{1 - \lambda}. \tag{5}
\]

Although we are primarily interested in the equilibrium values of the two aggregates, \(G\) and \(Y\), it will prove convenient to undertake some of the analysis in terms of the variables \(G\) and \(p = Y/G\). We write (4) as

\[
\tilde{g}_i = \frac{p - c_i}{p + a_i} G. \tag{6}
\]

Description of player \(i\)’s behavior must take into account the possibilities of corner solutions at which she chooses to allocate all of her endowment to either production or appropriation. Denote player \(i\)’s most preferred value of appropriation by \(\hat{g}_i\). Inspection of (6) confirms that, if \(p - c_i < 0\), the non-negativity constraint on \(\hat{g}_i\) is binding. Conversely, if the expression (6) exceeds \(e_i\), then player \(i\) will devote all her resources to appropriation. It can be readily confirmed that, if an interior solution exists, it satisfies the second-order condition associated with a payoff maximum.

A complete description of player \(i\)’s most preferred level of appropriation is given by

\[
\hat{g}_i = \rho_i(p, G) = \begin{cases} 
0 & \text{if } \tilde{g}_i \leq 0 \\
\tilde{g}_i & \text{if } \tilde{g}_i \in (0, e_i) \\
e_i & \text{if } \tilde{g}_i \geq e_i.
\end{cases} \tag{7}
\]
At any given allocation, depending on a player’s parameter values, she will belong to one of three groups—she may choose only to produce output, she may both produce and appropriate from others, or she may only appropriate output. Figure 1 shows, in \((p, G)\) space, into which group player \(i\) falls in terms of the prevailing values of the aggregate variables. It is worth drawing attention to some significant properties of this diagram. First, the boundary between allocations at which \(\hat{y}_i = 0\) and those at which \(0 < \hat{y}_i < e_i\), labeled \(i\), is linear—indeed, it is parallel to the \(G\) axis. Second, the boundary between the allocations at which \(\hat{y}_i = e_i\) and those at which \(0 < \hat{y}_i < e_i\), labeled \(i^+\), describes, with appropriate change of variables, the positive section of a rectangular hyperbola.\(^7\) Its asymptotes are the vertical line along which \(p = c_i\) and the horizontal dashed line along which \(G = e_i\). Clearly, for every player, \((p, G)\) space can be divided into the three regions, according to which group the player belongs to at the prevailing values of \(p\) and \(G\). Insofar as players’ endowments, productivities or aversions to thieving differ, the precise locations of these regions will vary across players. But, for each player, the qualitative features of the boundaries are as described above and shown in Figure 1.

[Insert Figure 1 here]

At any given allocation, the set \(I\) of all players can be decomposed into three mutually exclusive and exhaustive subsets: \(I = P \cup PA \cup A\), where \(P\) is the set of full-time producers, \(PA\) that of producer-appropriators, and \(A\) that of full-time appropriators. For each player, we will refer to these subsets as regimes.

### 4 Equilibrium

We now characterize the pure-strategy Nash equilibrium in terms of just two consistency conditions in the two unknowns, \(p\) and \(G\), instead of \(n\) best responses in \(n\) unknowns. The first consistency condition, which we label “appropriation consistency”, requires that the values of the aggregate variables, \(p\) and \(G\), be such that the sum of the implied most

---

\(^7\) This can be seen by noting that the condition that describes the boundary, \(\hat{y}_i = e_i\), can be written in the form \((p - c_i) (G - e_i) = k_i \equiv e_i (a_i + c_i)\): the product of the variables \((p - c_i)\) and \((G - e_i)\) equals a constant, \(k_i\).

---

9
preferred individual appropriation levels should be consistent with the observed aggregate level of appropriation, \( G \):

**Appropriation consistency.** The sum of most preferred individual levels of appropriation must equal the aggregate value of appropriation. Formally, this requires that any appropriation-consistent pair \((p^A, G^A)\) must satisfy the following relationship:

\[
\sum_{i \in I} \rho_i (p^A, G^A) = G^A. \tag{8}
\]

Given the partitioning of the players into the three groups, this consistency requirement can be written as

\[
\sum_{i \in PA} \frac{p_i^A - c_i}{p^A + a_i} + \sum_{i \in A} \frac{e_i}{G^A} = 1. \tag{9}
\]

Our second condition, that of “production consistency”, requires that the values of the aggregate variables, \( p \) and \( G \), be such that the implied sum of most preferred individual levels of production should be consistent with the observed aggregate output level, \( Y \). Player \( i \)'s most preferred level of production is \( \hat{y}_i = a_i \hat{x}_i = a_i (e_i - \hat{g}_i) = a_i (e_i - \rho_i (p, G)) \). Summing over all players and rearranging yields the following statement of production-consistency requirement:

**Production consistency.** At a production-consistent pair, \((p^P, G^P)\), the productivity-weighted sum of individual appropriation levels must equal that of endowments less total output:

\[
\sum_{i \in I} a_i \rho_i (p^P, G^P) = \sum_{i \in I} a_i e_i - Y^P. \tag{10}
\]

This requirement can alternatively be written as

\[
\sum_{i \in PA} \frac{a_i p_i^P - c_i}{p^P + a_i} + \sum_{i \in A} \frac{a_i e_i}{G^P} = \sum_{i \in I} \frac{a_i e_i}{G^P} - p^P. \tag{11}
\]

We now show that the qualitative properties of the two consistency conditions, (9) and (11), ensure that they possess a unique solution for the variables \( p \) and \( G \). This
immediately implies the existence of a unique Nash equilibrium in pure strategies.

**Lemma 1** The appropriation-consistent mapping, $\Gamma^A : p \rightarrow G$, has a closed graph and is everywhere non-decreasing.

**Lemma 2** The production-consistent mapping, $\Gamma^P : p \rightarrow G$, has a closed graph and is everywhere strictly decreasing.

**Proposition 1** Fulfilment of both appropriation and production consistencies is necessary and sufficient for the existence of a unique pure-strategy Nash equilibrium, $(p^*, G^*)$, regardless of how many players there are and regardless of the extent to which these players may differ from one another in terms of endowments, productivities and costs of appropriation.

The rest of this section proves these assertions.

### 4.1 Appropriation-consistency mapping

Recall the appropriation-consistency requirement (9) where the assignment of players to the sets $PA$ and $A$ at any point in $(p, G)$ space is itself endogenous. Figure 2 shows the boundaries between the three regimes for each player in a three-player example. The aggregate endowment of the economy implies that only allocations represented by points on or below the dotted line corresponding to $G = E \equiv \sum_{j \in I} e_j$ are feasible. The lines labeled $i (= 1, 2, 3)$ show the boundary allocations at which player $i$ prefers pure production to production-appropriation. Those labeled $i^+$ show the boundary allocations at which player $i$ prefers pure appropriation to production-appropriation. The locations of each of these boundaries are determined by the exogenous parameters within the model.

[Insert Figure 2 here]

These boundaries divide $(p, G)$ space into a number of regions, each corresponding to a given assignment of all players to one of these three sets. For example, at the allocation represented by the point $\alpha$—indeed, at all allocations within the lightly shaded area
marked \( A = I \)—all three players would choose to be pure appropriators. At \( \beta \), player 3 would choose to both produce and appropriate, while players 1 and 2 would choose pure appropriation. At \( \gamma \), player 2 would choose pure appropriation while players 1 and 3 would choose to produce and appropriate. Finally, at \( \delta \), player 3 would only produce, while players 1 and 2 would both produce and appropriate.

It is worth drawing attention to three types of region that are of particular interest.

1. Regions within which \( A = \emptyset \). If an appropriation-consistent allocation lies within such a region, then in the neighborhood of any such allocation the requirement becomes

\[
\sum_{i \in PA} \frac{p^A - c_i}{p^A + a_i} = 1.
\]

Suppose, for a moment, that we are looking at an allocation within such a region that satisfies the appropriation-consistency condition. Then the form taken by the condition uniquely determines the required value of \( p \), since \( G \) simply does not appear. Within this region, the graph consists of a vertical segment corresponding to the unique value of \( p \) that satisfies this condition.

2. Regions within which \( PA = \emptyset \). If an appropriation-consistent allocation lies within a region in which \( PA = \emptyset \), then in the neighborhood of any such allocation the requirement becomes

\[
\sum_{i \in A} \frac{e_i}{G^A} = 1.
\]

This uniquely determines the required value of \( G \), and the graph within such a region is a horizontal line along which \( G = \sum_{i \in A} e_i \). Note further that, if the set \( P \) is also empty, so that \( A = I \), then the condition implies that \( G = E \).

3. Regions within which both \( A \) and \( PA \) are non-empty. Within such a region, both of the summation terms appear on the left hand side of the consistency condition. Inspection makes it clear that, starting from an appropriation-consistent allocation in such a region, an increase in \( p \) raises the term \( \sum_{i \in PA} \frac{p^A - c_i}{p^A + a_i} \) and must therefore be accompanied by an increase in \( G \) in order to maintain appropriation consistency.
Armed with these observations, it is an easy matter to piece together the graph of allocations that satisfy the appropriation-consistency requirement. First, consider the lightly shaded set in the upper right-hand portion of Figure 3 where \( A = I \). This must clearly be non-empty. Points within this set on the line along which \( G = E \) certainly satisfy the appropriation-consistency condition. Moreover, they are the only allocations within this set that do so. At such points, every player chooses to devote her entire endowment to appropriation, and the aggregate level of appropriation must equal the aggregate endowment. This set is represented by the thick horizontal line running from the point \((\bar{p}, E)\) to \((\infty, E)\).

[Insert Figure 3 here]

Starting at the point \((\bar{p}, E)\), where player 3 is on the boundary between being a pure appropriator and a producer-appropriator, reduce \( p \) by a small amount. Player 3 is no longer a pure appropriator, but is a producer-appropriator. Since the sets \( A \) and \( PA \) each now have at least one element, inspection of (9) shows that further decline in \( p \) must be accompanied by a decline in \( G \) to maintain appropriation consistency. The graph of appropriation-consistent allocations is strictly upward-sloping through this region. Eventually, it meets the boundary labeled \( 2^+ \). At this point, player 2 switches from pure appropriation to production-appropriation. The graph continues to be upward-sloping. At last, it meets the boundary labeled \( 1^+ \), at which player 1 switches to production-appropriation. At this point, the set \( A \) becomes empty, and the appropriation-consistency requirement becomes \( \sum_{i \in PA} \frac{pA - ci}{pA + ai} = 1 \), which uniquely determines the value of \( p \) at \( p^* \).

The qualitative properties of the appropriation-consistent graph (the thick continuous line in Figure 3) are robust in the sense that they are true not just of our three-player example, but more generally. Let us look at two other examples that exhibit features, not shown in Figure 3, that may characterize the appropriation-consistent graph.

---

\( ^8 \) This feature depends critically on the linear production technology. If production is nonlinear, \( G \) will appear in the appropriation-consistency condition even if all players are producer-appropriators. Note that the variable \( p \equiv Y/G \) is not well defined when \( G = 0 \). This does not threaten our subsequent analysis, since \( G = 0 \) can never hold at an equilibrium of this game.
First note that, if all players were identical, then at the point \((\bar{p}, E)\), all would switch from pure appropriation to production-appropriation in response to a further fall in \(p\). In this case, the graph immediately becomes vertical, since the set \(A\) is empty and the consistency condition takes the form \(\sum_{i \in I} \frac{p - c_i}{p + a_i} = 1\). This uniquely determines the value \(\bar{p}\). In this situation, there is no strictly upward-sloping segment of the graph. It takes the form shown in Panel (a) of Figure 4, and \(\bar{p} = p\).

Second, it is possible that, as we reduce \(p\) along a strictly upward-sloping segment of the graph, an allocation is reached at which all existing producer-appropriators switch to pure production, leaving a non-empty set of pure appropriators. This happens in Panel (b) of Figure 4, and produces a horizontal step in the graph at a value of \(G\) that is strictly less than \(E\). In this example, players 1 and 2 are pure appropriators and player 3 is a pure producer within the relevant region.

Neither of these possibilities destroys the continuity and slope properties of the graph. To summarize, the appropriation-consistency condition implies a mapping from \(p \in [\bar{p}, \infty)\) to \(G \in [0, E]\), which we denote by \(\Gamma^A(p)\), that has a closed graph and is everywhere non-decreasing.

### 4.2 Production-consistency mapping

The graph implied by the production-consistency requirement (11) can be traced out simply. Note three features of this condition. First, even if either \(PA\) or \(A\) is empty, so that one or the other of the summation terms disappears from the left hand side (LHS), the condition still contains both \(p^P\) and \(G^P\). Second, if \(G\) alone increases, the right hand side (RHS) will fall by at least as much as the LHS because the relevant summation term can be written as \(\sum_{i \in I} \frac{a_i e_i}{p^P} = \sum_{i \in A} \frac{a_i e_i}{G^P} + \sum_{i \in A} \frac{a_i e_i}{G^P}\). Finally, an increase in \(p\) alone leads to a fall in the RHS, and an increase in the LHS, of (11). These observations imply that the graph of this condition is everywhere strictly downward-sloping.

The production-consistency condition implies a single-valued mapping from \(p \in [p, \infty)\) to \(G \in [0, E]\), where \(\frac{1}{p} < \frac{1}{\bar{p}}\). This last inequality can be understood by noting that, at
any point within the lightly shaded area of Figure 5, all players are pure appropriators. This suggests that (11) requires \( p^P = 0 \) in the area. However, that area requires \( p > 0 \). Thus, (11) never holds in the area. This guarantees that \( p < \bar{p} \). In short, the production-consistency mapping, denoted by \( \Gamma^P (p) \), has a closed graph and is everywhere strictly decreasing.

[Insert Figure 5 here]

The thick continuous lines in Figure 5 illustrate the graphs of \( \Gamma^A (p) \) and \( \Gamma^P (p) \). Their continuity, slope and mapping-range properties ensure a unique intersection, that determines the equilibrium values of \( p \) and \( G \).

### 4.3 Equilibrium structure

Before undertaking comparative static analysis of the unique equilibrium, we draw attention to a few features of a given equilibrium. Our formulation, by using the same aggregates as arguments in every player’s payoff and behavioral functions, makes it easy to read off how differences in parameters across players map into differences in their equilibrium behavior and payoffs. Recall that player \( i \)’s payoff can be described by

\[
v_i (\hat{g}_i, p, G) = \lambda a_i (e_i - \hat{g}_i) + (1 - \lambda) p\hat{g}_i - b_i\hat{g}_i \tag{12}
\]

where \( \hat{g}_i = \rho_i (p, G) \) as in (7). At a given allocation, the values of a player’s parameters determine to which of the three activity sets she belongs. Player \( i \)’s equilibrium payoff is

\[
v_i^* = \begin{cases} 
[(1 - \lambda) p^* - b_i] e_i & \text{for } i \in A \\
\lambda a_i e_i + \frac{(1 - \lambda)(p^* - a_i)^2 G^*}{p^* + a_i} & \text{for } i \in PA \\
\lambda a_i e_i & \text{for } i \in P. 
\end{cases} \tag{13}
\]

Three implications can be drawn. First, a player who chooses pure appropriation at an allocation is unaffected by changes in \( G \) in the neighborhood of that allocation. Her indifference curve is locally vertical in \((p, G)\) space. Second, a player who chooses pure
production is not affected by changes in either of the endogenous aggregates. Matters are a little more complicated for a producer-appropriator. However, the following lemma can be readily established:

**Lemma 3** Suppose that players \( j \) and \( k \) are both producer-appropriators at \((p,G)\), and that \( b_j = b_k \). Then, if \( a_j > a_k \), player \( j \)'s indifference curve through \((p,G)\) is steeper than that of player \( k \).

We will exploit this observation later when we infer changes in existing players’ equilibrium payoffs in response to the exogenous addition of one more player to the game.

**4.3.1 Relationship between endowments and payoffs**

Our model is closely related to Hirshleifer (1991). He used a two-player model to show that the introduction of appropriation as the sole means of securing consumption reduces the initial wealth gap in Nash equilibrium. By setting \( s = b_i = 0 \) and \( a_i = a \in (0, \infty) \) for all \( i \), our model becomes a special case of his model (i.e. \( s = m = 1 \) in his model) where all output are subject to appropriation. Using this reduced form, we can easily confirm that his paradox-of-power theorem holds in the case of many players.\(^9\)

Besides, we can show that when each player’s output is not totally subject to appropriation, i.e. \( \lambda > 0 \), the paradox of power still holds but only in weak form, provided that not all players under comparison are pure producers. We examine how player \( i \)'s endowment influences her payoff by comparing the equilibrium payoffs across a subset of players who have identical productivity and theft aversion parameters, but who enjoy different endowments. Denote the common parameter values by the unindexed quantities \( a, b, \) and \( c = \frac{a + b}{1 - \lambda} \).

**Proposition 2** Suppose \( \lambda > 0 \), and players \( j \) and \( k \) are identical except that \( e_j > e_k \). If \( p^* > c \), then the ratio of \( j \)'s endowment to \( k \)'s is greater than the ratio of their equilibrium payoffs, which in turn is greater than one, i.e. \( e_j/e_k > v_j^* / v_k^* > 1 \).

---

\(^9\)Due to the limited space, we focus on the reproduction of his paradoxes for generalization. See Hirshleifer (1991: 185-8) for the precise definitions and intuitions behind the strong and weak paradoxes of power.
If \( p^* > c \), the value of \( i \)'s endowment determines whether she is a producer-appropriator or a pure appropriator at equilibrium:

\[
e_i > \frac{p^*-c}{p^*+a} G^* \implies \hat{g}_i = \frac{p^*-c}{p^*+a} G^*
\]

\[
\implies v_i^* = \lambda a \left[ e_i - \frac{p^*-c}{p^*+a} G^* \right] + (1 - \lambda) \frac{p^*-c}{p^*+a} p^* G^r,
\]

\[
e_i \leq \frac{p^*-c}{p^*+a} G^*
\]

\[
\implies \hat{g}_i = e_i
\]

\[
\implies v_i^* = [(1 - \lambda) p^* - b] e_i.
\]

Two possibilities arise. One is that \( j \) becomes a producer-appropriator while \( k \) becomes a pure appropriator, i.e.

\[
e_j > \frac{p^*-c}{p^*+a} G^* \geq e_k.
\]

After a few manipulations, \( e_j/e_k > v_j^*/v_k^* \) reduces to \( e_j > \frac{p^*-c}{p^*+a} G^* \) which is currently assumed to hold. Also we can write \( v_j^*/v_k^* > 1 \) as

\[
\lambda a \left[ e_j - \frac{p^*-c}{p^*+a} G^* \right] > 0 \geq [(1 - \lambda) p^* - b] \left[ e_k - \frac{p^*-c}{p^*+a} G^* \right].
\]

The second possibility is that both players become producer-appropriators, i.e. both endowments are greater than \( \frac{p^*-c}{p^*+a} G^* \). It is easy to show that \( e_j/e_k > v_j^*/v_k^* > 1 \) holds also in this case.

The proof under the first possibility above generalizes Hirshleifer's (1991) weak paradox of power to cases of \( \lambda > 0 \). Under the second possibility, he showed \( e_j/e_k > v_j^*/v_k^* = 1 \), which he called the strong paradox of power. We have shown that only \( \lambda = 0 \) yields the strong form, and that the weak form is more generally applicable.

If \( p^* \leq c \) instead, all players within the subset will choose \( \hat{g}_i = 0 \), i.e. they will become pure producers, regardless of player \( i \)'s endowment. The realized payoff will then be \( v_i^* = \lambda a e_i \), and hence \( e_j/e_k = v_j^*/v_k^* > 1 \). Thus, within a subset of pure producers, the paradox of power does not hold even in weak form. This possibility has not been mentioned in the literature, as it does not arise under \( \lambda = 0 \).
4.3.2 Relationship between productivities and payoffs

Using a two-player model like Hirshleifer’s (1991), Garfinkel and Skaperdas (2007: 666) show that in equilibrium more productive players are strictly worse off than less productive players. Again, by setting \( \lambda = 0 \) and comparing the equilibrium payoffs across a subset of players whose endowments and attitudes towards thievery are identical, we can reproduce their result that \( v_j^* < v_k^* \) if \( a_j > a_k \).

Our model additionally suggests that, if \( \lambda > 0 \), their result does not hold when \( k \)’s productivity is sufficiently high.

**Proposition 3** Suppose \( \lambda > 0 \), and players \( j \) and \( k \) are identical except that \( a_j > a_k \).

Then, the ratio of \( j \)'s productivity to \( k \)'s is greater than the ratio of their equilibrium payoffs, i.e. \( a_j / a_k > v_j^* / v_k^* \), if \( a_k < \frac{(1-\lambda) p^* - b}{\lambda} \), that is, if \( k \) is not a pure producer in equilibrium.

Inspection of (13) implies that a pure appropriator’s equilibrium payoff is independent of her productivity while a pure producer’s equilibrium payoff is increasing in her productivity at a constant rate, \( \lambda e \). For a producer-appropriator, we have

\[
\frac{\partial v_i^*_{PA}}{\partial a_i} = \lambda e - \frac{(1 - \lambda) \left( p^* - \frac{\lambda a_i + b}{1 - \lambda} \right)}{p^* + a_i} G^* \left( \frac{2\lambda}{1 - \lambda} + \frac{p^* - \frac{\lambda a_i + b}{1 - \lambda}}{p^* + a_i} \right),
\]

where the second term is decreasing in \( a_i \). Since player \( i \) chooses to be a pure appropriator as long as \( \frac{p^* - \frac{\lambda a_i + b}{1 - \lambda}}{p^* + a_i} G^* \geq e \), the value of \( a_i \) at which the player is on the border between pure appropriation and production-appropriation is given by

\[
\frac{(1 - \lambda) p^* \left( 1 - \frac{e}{\sigma^*} \right) - b}{\lambda + (1 - \lambda) \frac{e}{\sigma^*}},
\]

which we can substitute into the second term of the first derivative expression in order to confirm that the equilibrium payoff is initially decreasing in \( a_i \) as she switches from pure appropriation to production-appropriation. It is easy to confirm that the second derivative is positive. As we increase \( a_i \) further, \( v_i^*_{PA} \) achieves the minimum and then
rises at an increasing rate until $p^* = c_i$ or equivalently

$$a_i = \frac{(1 - \lambda) p^* - b}{\lambda}$$

at which she switches from production-appropriation to pure production. Also note that at this value of $a_i$ we have $v_{i \in A}^* = v_{i \in P}^*$, as we can verify by inspecting (13).

If both $j$ and $k$ were pure appropriators, their equilibrium payoffs would be the same. Hence we have $a_j/a_k > v_{j \in A}^*/v_{k \in A}^* = 1$. If both were producer-appropriators instead, we can readily confirm $a_j/a_k > v_{j \in P_A}^*/v_{k \in P_A}^*$ because this inequality is equivalent to

$$\frac{a_j p^* + a_j}{a_k p^* + a_k} > 1 > \left( \frac{(1 - \lambda) p^* - b - \lambda a_j}{(1 - \lambda) p^* - b - \lambda a_k} \right)^2,$$

which is true. These in turn implies that $a_j/a_k > v_{j \in P_A}^*/v_{k \in A}^*$ because $a_j < \frac{(1-\lambda)p^*-b}{\lambda}$ if $j$ is a producer-appropriator, which suggests $v_{j \in P_A}^* < v_{j \in A}^* = v_{k \in A}^*$. Since clearly $v_{j \in P}^* = \lambda e a_j > v_{k \in P}^* = \lambda e a_k$ if both players are pure producers, we finally check the case where $j$ is a pure producer and $k$ a producer-appropriator. To hold $a_j/a_k > v_{j \in P}^*/v_{k \in P_A}^*$ is equivalent to holding

$$a_k < \frac{(1 - \lambda) p^* - b}{\lambda},$$

which simply means that, as long as $k$ is indeed a producer-appropriator at a given value of $a_k$, the inequality holds. In other words, the inequality is violated if $k$’s productivity is high enough to induce her to become a pure producer.\footnote{A diagrammatic illustration is provided in Cornes et al. (2010: Figure 7).} Again, this possibility does not arise under $\lambda = 0$.

5 Comparative statics

Heterogeneity has several notable implications. First, it introduces the realistic possibility that, at a given equilibrium, different players are in different regimes—pure appropriators, producer-appropriators and pure producers may all coexist in equilibrium. Second, even within the set, say, of producer-appropriators, differences in the parameters across those
players may lead to different behavioral and welfare responses to a given exogenous shock. Finally, some exogenous shocks may well provoke endogenous changes in the assignment of players to the sets \((A, PA, P)\).

This section is intended to be illustrative rather than exhaustive. We concentrate on the implications of the exogenous addition of a player to the game, as this is a question that could not be examined in the presence of many heterogeneous players.\(^{11}\) Our method of analysis carries over to other comparative static problems, which we report in Cornes et al. (2010).

5.1 Exogenous entry of an additional player

The addition of a player, characterized by the trio of parameters \((a_{n+1}, b_{n+1}, e_{n+1})\), shifts both consistency graphs. These shifts affect the intersection of the graphs, and hence the equilibrium \((p^*, G^*)\). Consider first the consequences of adding a player for the appropriation-consistency condition.

**Lemma 4** Addition of a player to the game shifts \(\Gamma^A(p)\) up or to the left along the range of allocations \((p, \Gamma^A(p))\) at which the added player would choose to be either a pure appropriator or a producer-appropriator. The graph remains the same along the range where she would choose to be a pure producer.

At allocations \((p^A, G^A)\) at which the entrant would prefer to be a pure producer, her entry does not affect the LHS of (9), and therefore does not shift the appropriation-consistency graph. However, at allocations at which she would prefer to be either a producer-appropriator or a pure appropriator, her entry adds a positive term to the LHS of (9). Thus, either \(p\) must fall or \(G\) must rise to maintain appropriation consistency. In short, depending on which regime the new player would choose at the initial allocation, the appropriation-consistency graph either remains unaffected or shifts upward or to the left. Now, consider the production-consistency condition.

\(^{11}\)We refer the reader to Cornes et al. (2010: 17-9) for the comparative statics in the case of identical players, since those results can easily be obtained without the two-aggregate method. There are typos in the summary table presented there. The four endogenous variables in the first row are equilibrium values and hence should be starred. Also, \(e\) has no effect on \(p^*\), as is clear from their Eq. (15).
**Lemma 5** Addition of a player to the game shifts $\Gamma^p(p)$ up to the right along the range of allocations $(p, \Gamma^p(p))$ at which the added player would choose to be either a producer-appropriator or a pure producer. The graph remains the same along the range where she would choose to be a pure appropriator.

At allocations $(p^P, G^P)$ at which the new player would prefer to be a pure appropriator, her entry adds the same term, $\frac{e_{n+1}}{G_{n+1}}$, to both sides of (11). This does not affect the production-consistency condition and does not shift the associated graph. If, at $(p^P, G^P)$, she prefers to be a producer-appropriator, her entry again adds a term to each side of (11). However, the fact that she prefers to be a producer-appropriator implies that $\frac{p^P - c_{n+1}}{p^P + a_{n+1}} < \frac{e_{n+1}}{G_{n+1}}$. Thus, without further adjustments, production consistency is violated, and either $p$ or $G$ must rise to restore production consistency. The production consistency graph must shift upward to the right. Finally, the entry of a player who prefers pure production at a point $(p^P, G^P)$ increases the RHS of (11), but leaves the LHS unaffected. Again, the production consistency graph must shift upward to the right in response to her entry.

[Insert Figure 6 here]

Let us illustrate these qualitative observations, using three numerical examples. In all three panels of Figure 6, the thick continuous lines $\Gamma^A_0(p)$ and $\Gamma^P_0(p)$ are, respectively, the same appropriation- and production-consistency graphs associated with a three-player game. They are consistent with the following parameter values: $\lambda = 0$, $(b_1, b_2, b_3) = (0, 0, 0)$, $(a_1, a_2, a_3) = (1, 2, 4)$, and $(e_1, e_2, e_3) = (4, 1, 2)$. The intersection of the graphs identifies the initial equilibrium, $(p_0^A, G_0^A) \simeq (1.203, 4.489)$, and hence $Y^A \simeq 5.399$. Players 1 and 3 are producer-appropriators, and player 2 a pure appropriator in equilibrium. That is, $(g_1^0, g_2^0, g_3^0) \simeq (2.451, 1, 1.038)$ and $(y_1^0, y_2^0, y_3^0) \simeq (1.549, 0, 3.849)$. Their equilibrium payoffs are $(v_1^0, v_2^0, v_3^0) \simeq (2.948, 1.203, 1.248)$. Each panel shows shifts of the graphs implied by the entry of a fourth player.
5.2 When the entrant is a pure producer at the initial equilibrium

In Panel (a), the new player is described by the parameter values \((a_4, b_4, e_4) = (1, 1.3, 1)\). The thin dashed lines are the boundaries between the three regimes for her. In the situation depicted, her aversion to the act of thieving would lead her to choose pure production in the neighborhood of the initial equilibrium. Her entry does not shift the appropriation-consistency graph in that neighborhood (Lemma 4), but it does shift the production consistency graph upward to the right (Lemma 5). The thick dashed lines illustrate the changes. As a consequence of the entry, both \(p^*\) and \(G^*\), and therefore \(Y^*\), must rise. At the new equilibrium, \((p^1, G^1) \simeq (1.238, 4.750)\), and hence \(Y^1 \simeq 5.889\).

**Proposition 4** The addition of an extra player who would choose pure production at the pre-entry equilibrium raises the equilibrium values of both \(p\) and \(G\), and hence \(Y\). It increases the equilibrium payoffs of all existing pure appropriators and producer-appropriators. All existing producer-appropriators increase appropriation effort and reduce production. Furthermore, an existing pure producer may become a producer-appropriator or a pure appropriator at the new equilibrium. If she does, her payoff increases. If she does not, her payoff remains the same.

An increase in the equilibrium payoff across producer-appropriators and pure appropriators is implied by (13). Note that this increase takes place whether or not the added player is a pure producer at the post-entry equilibrium, as is clear in Figure 6(a). The potential group reassignment of existing pure producers is implied by (6) and (7). These two expressions also suggest the possibility that, as \(\tilde{g}_i\) is increased, a producer-appropriator at the initial equilibrium may become a pure appropriator at the new equilibrium. In the current example, however, both players 1 and 3 remain to be producer-appropriators, i.e. \((g^1_1, g^1_2, g^1_3, g^1_4) \simeq (2.628, 1, 1.123, 0)\). In any case, the existing players would not oppose to the arrival of a player who would choose pure production at \((p^0, G^0)\).
5.3 When the entrant is a pure appropriator at the initial equilibrium

In Panel (b), the added player is described by \((a_4, b_4, c_4) = \(1, 0.5, 1\)). She chooses pure appropriation in the neighborhood of the initial equilibrium. Her arrival provokes a shift upward and to the left of the appropriation-consistency graph (Lemma 4), but does not disturb the production-consistency condition in the neighborhood of the initial equilibrium (Lemma 5). Thus, the entry of a player who would choose pure appropriation at \((p^0, G^0)\) leads to a fall in \(p^*\) and an increase in \(G^*\). At the new equilibrium, \((p^1, G^1) \simeq (0.889, 5.752)\), and hence \(Y^1 \simeq 5.111\). Thus, in this example, the total output is smaller at the new equilibrium. However, this is not always the case because some players increase appropriation activity while some others may decrease it and increase production. This is evident from (6) and (7) which suggest that a sufficient fall in \(p^*\), i.e. \(p^* \leq c_i\), can turn an existing producer-appropriator or a pure appropriator into a pure producer. In this example, however, both players 1 and 3 increase appropriation activity, and player 2 remains to be a pure appropriator, i.e. \((g_1^1, g_2^1, g_3^1, g_4^1) \simeq (2.706, 1, 1.046, 1)\).

**Proposition 5** The addition of an extra player who would choose pure appropriation at the initial equilibrium lowers \(p^*\) and raises \(G^*\). Existing pure appropriators are made worse off and may also become producer-appropriators or pure producers at the new equilibrium. Existing pure producers are unaffected.

Again, (6), (7) and (13) confirm this assertion. The effects on the equilibrium resource allocation and payoffs of existing producer-appropriators are ambiguous. They can be made either better or worse off. Lemma 3 implies that, assuming the appropriation cost parameter does not differ much across the existing producer-appropriators, less productive producer-appropriators who exhibit flatter indifference curves are more likely to be made better off when \(G^*\) rises and \(p^*\) falls. More productive producer-appropriators may react to the addition of a pure appropriator by reallocating their endowed resources from appropriation to production. On the other hand, less productive producer-appropriators would make more effort to appropriate than before. In our numerical example, however,
both players 1 and 3 do not have sufficiently steep indifference curves and are made worse off by the entrant, i.e. \((v_1^1, v_2^1, v_3^1, v_2^1) \simeq (2.405, 0.889, 0.929, 0.389)\).

### 5.4 When the entrant is a producer-appropriator at the initial equilibrium

In Panel (c), which is consistent with \((a_4, b_4, e_4) = (2.6, 0, 3.5)\), the new player would choose production-appropriation at \((p^0, G^0)\). The entrant increases the LHS of (9), leading to an upward shift of the appropriation-consistency graph in the neighborhood of the initial equilibrium (Lemma 4). At the same time, the production-consistency graph shifts to the right (Lemma 5). The qualitative nature of the shifts in this example is such that, although \(G^*\) unambiguously rises, \(p^*\) may either rise or fall, depending on the slopes of the graphs and the magnitudes of their shifts. If the added player increases \(p^*\) in the neighborhood of \((p^0, G^0)\), we have the same situation as in the case of adding a player who would choose pure production at the pre-entry equilibrium (Proposition 4). If \(p^*\) decreases instead, the situation is simialr to the case of adding a player who would become a pure appropriator at \((p^0, G^0)\) in Proposition 5.

Our example here illustrates this latter situation in which the equilibrium value of \(p\) falls as a result of the addition. Indeed, we have chosen parameter values to show the possibility that was mentioned in the previous subsection but was not illustrated by the numerical example there. The post-entry equilibrium is at \((p^1, G^1) \simeq (0.842, 8.023)\) and \(Y^1 \simeq 6.753\). As a result, \((g_1^1, g_2^1, g_3^1, g_4^1) \simeq (3.667, 1.1.395, 1.962)\) and \((v_1^1, v_2^1, v_3^1, v_2^1) \simeq (3.056, 0.842, 1.174, 1.651)\). Both existing producer-appropriators increase appropriation activity. Player 1 who is the less productive and more endowed of the two initial producer-appropriators is better off while the other is worse off at the new equilibrium.

### 6 Concluding remarks

There are many applications that could lend themselves naturally to our approach through the use of two aggregates. As we acknowledged in the introduction, the analysis of market
games has already benefitted from the two-aggregate method. Natural extensions to this line of work involve the further examination of timing issues, and incorporation of the possibility that each player has endowments of more than one good, so that the model can endogenously determine which good the player supplies to, and which she demands from, the market.

An enhanced understanding of market models is relevant to attempts to understand the implications of the creation of markets in tradable rights to pollute. For example, consider an oligopolistic industry in which production also generates pollution as a by-product, or else depletes a scarce open-access resource. If firms differ with respect to their roles as polluters or depleters, then again a two-aggregate structure naturally emerges.

Our approach may also prove useful for the analysis of output sharing and cost sharing. Suppose, for example, a group of players, each with her own idiosyncratic innate productivity, are involved in a joint productive enterprise. Then the aggregate output which will be shared amongst them will depend upon the sum of their effective inputs. In addition, if the agreed rule for sharing that output depends in part on the observed number of hours that each provides, then the total number of hours—which will be a weighted sum of their effective input levels—will also be an argument of individual payoff and behavioral functions.

Many further potential applications to models involving many heterogeneous players spring readily to mind, in which the present approach avoids the limitations imposed by the proliferation of dimensions when the best response formulation is used. In short, this article is no more than a preliminary step in exploring the usefulness of studying games with more than a single aggregate, and an invitation to others to explore our method further. However, it is a promising step.

References


Figure 6

(a) Entry of a ‘pure producer’

(b) Entry of a ‘pure appropriator’

(c) Entry of a ‘producer-appropriator’