WORKING PAPERS IN ECONOMICS & ECONOMETRICS

FULLY AGGREGATIVE GAMES

RICHARD CORNES
Economics Program
Research School of Social Sciences
Coombs Building
Australian National University
Canberra
ACT 0200
Australia

ROGER HARTLEY
Economics
School of Social Sciences
University of Manchester
Oxford Road, Manchester
M13 9PL, UK

August 7, 2009

JEL codes: C72

Working Paper No: 505
ISBN: 086831 505 2

Abstract

A game is fully aggregative if payoffs and marginal payoffs depend only on a player's own strategy and a function of the strategy profile which is common to all players. We characterize the form which this function must take in such a game and show that the game will be strategically equivalent to another game in which the function is the simple sum of strategies.

JEL classification: C72.

Keywords: Noncooperative game theory, aggregative games.

Corresponding author: Richard Cornes, Economics Program, Research School of Social Sciences, Coombs Building, Australian National University, Canberra, ACT 0200, Australia.

ph: +61 02 6125 4605

email: rccornes@aol.com

Acknowledgement : Much of the first author's work on this paper was undertaken while he was a Visiting Scholar at the Center for Economic Studies, University of Munich. The support of the Center is gratefully acknowledged. Later stages were facilitated by the award of a Hallsworth Visiting Research Fellowship at the University of Manchester.

1 Introduction

Consider a simultaneous move game Γ with n players and, for $i = 1, \ldots, n$, let $X_i \subseteq \mathbb{R}$ denote the strategy set and $\pi_i : \mathbf{X} = \prod_{j=1}^{n} X_j \to \mathbb{R}$ the payoff function of player i. Such a game is called *aggregative* if there exist continuously differentiable functions $t : \mathbf{X} \longrightarrow \mathbb{R}$ and $\nu_i : \mathbb{R}^2 \to \mathbb{R}$ such that, for all $i = 1, \ldots, n$ and all $\mathbf{x} \in \mathbf{X}$

$$\pi_i(\mathbf{x}) = \nu_i(x_i, t(\mathbf{x})).$$

We will refer to the function t as the *aggregator.* Many games exhibit this structure. Examples include Cournot oligopoly, the voluntary contribution model of public good provision, contest theory and open access resource exploitation.

When studying Nash equilibria of such games, a particularly fruitful way of exploiting the aggregative structure can be employed when marginal payoffs also depend only on own strategy and the aggregator. We call such games *fully aggregative.* Every player in such a game has a *replacement correspondence.* For any i and $\tau \in t(\mathbf{X})$, the replacement correspondence R_i of player i is the set of strategies which i will play in any equilibrium $\mathbf{x}$ satisfying $t(\mathbf{x}) = \tau$. Thus $R_i(\tau)$ is the set of $x_i \in X_i$ which are best responses to every $\mathbf{x}_{-i} \in \prod_{j \neq i} X_j$ such that $t(x_i, \mathbf{x}_{-i}) = \tau$. Then, $\widehat{\mathbf{x}} \in \mathbf{X}$ is a pure-strategy Nash equilibrium if and only if $\widehat{x}_i \in R_i(t(\widehat{\mathbf{x}}))$ for all $i \in I$. When $R_i(\tau)$ is a singleton $\{r_i(\tau)\}$ for all i and all τ, this means $\widehat{\tau}$ is an equilibrium value of the aggregator if and only if $\widehat{\tau}$ is a fixed point of the mapping $\tau \longmapsto t(r_1(\tau), \ldots, r_n(\tau))$. These observations imply that equilibria are characterized by fixed points in one dimension as opposed to the n-dimensional mappings of conventional best responses. This insight been exploited by many authors to study existence, uniqueness and comparative statics of equilibria.

Selten [16] suggested a way of exploiting the aggregative structure to establish equilibrium existence in a Cournot oligopoly model[1]. His approach was exploited and developed by Novshek [12] and Kukushkin [11]. It was used by Okuguchi [14] and analyzed in a more general setting by Corchon ([1], [2]). Most recently, Cornes and Hartley develop this approach more

[1] Selten [16] uses the term *einpassungsfunktion*, which Phlips [15] translates as "fitting-in function." Novshek and Sonnenschein [13] refer to this relationship as the "backward map". Novshek [12] refers to it as the "backwards reaction mapping".

Selten's suggestion is also briefly noted and exploited by Friedman [9], Phlips [15], Wolfstetter [20] and Vives [19], but none of these authors develop the idea further.

systematically and apply it to a range of applications, including public good provision [7], contest theory [5] and surplus sharing [4], of which the tragedy of the commons is an example.

2 Aggregators for fully aggregative games

Since

$$\frac{\partial}{\partial x_i}\pi_i(\mathbf{x}) = \nu_{i1}(x_i, t(\mathbf{x})) + \nu_{i2}(x_i, t(\mathbf{x}))\frac{\partial}{\partial x_i}t(\mathbf{x}),$$

a sufficient and (unless ν_{i2} vanishes) necessary condition for a game to be fully aggregative is that $\partial t/\partial x_i$ is a function of x_i and t alone. The aggregator will clearly have such a property if there exist continuously differentiable functions $H : \tau(\mathbf{X}) \to \mathbb{R}$ and $F_i : X_i \to \mathbb{R}$ for all $i \in I$, such that H is strictly increasing and

$$t(\mathbf{x}) = H^{-1}\left[\sum_{j=1}^{n} F_j(x_j)\right] \text{ for all } \mathbf{x} \in \mathbf{X}. \tag{1}$$

In this paper, we show that, under appropriate smoothness and monotonicity conditions, the converse is true. If $\partial t/\partial x_i$ is a function of x_i and t alone, then t must satisfy (1). This functional form also implies that Γ is strategically equivalent to a game with the even simpler aggregator: $t = \sum_{j\in I} x_j$.

Definition 1 *The sufficient statistic $t(\mathbf{x})$ is said to be* ***regular*** *if and only if it is twice continuously differentiable and all first partial derivatives are positive for all $\mathbf{x} \in \mathbf{X}$.*

Note that the definition really only requires all first partial derivatives to have the same sign. If these were negative, we could satisfy the definition by replacing t with $-t$ and redefining the function $\upsilon_i(\cdot)$. Armed with this definition, we can now state our central proposition.

Proposition 1 *Suppose that $t(\mathbf{x})$ is regular and $\partial t/\partial x_i$ is a function of x_i and t alone for all $i = 1, \ldots, n$ and all all $\mathbf{x} \in \mathbf{X}$. If $n \geq 3$, (1) is satisfied with H and F_i strictly increasing functions for all i.*

Proof. Fix i. By hypothesis, there exists a function $\phi_i(x_i, \tau)$ such that

$$\frac{\partial t(\mathbf{x})}{\partial x_i} = \phi_i(x_i, t(\mathbf{x})), \tag{2}$$

for all $\mathbf{x} \in \mathbf{X}$. The proof proceeds by finding the general solution to these equations via a process of successive refinement of the functions $\phi_i(\cdot)$. We start by showing that they are separable.

First, observe that, for any $j \neq i$,

$$\frac{\partial^2 t(\mathbf{x})}{\partial x_j \partial x_i} = \frac{\partial \phi_i(x_i, t(\mathbf{x}))}{\partial \tau} \frac{\partial t(\mathbf{x})}{\partial x_j} = \frac{\partial \phi_i(x_i, t(\mathbf{x}))}{\partial \tau} \phi_j(x_j, t(\mathbf{x})). \tag{3}$$

Regularity implies that the right-hand side of (3) is symmetric in i and j and therefore that

$$\psi_i(x_i, t(\mathbf{x})) = \psi_j(x_j, t(\mathbf{x})), \tag{4}$$

for all j and all $\mathbf{x} \in \mathbf{X}$, where

$$\psi_i(x, \tau) = \frac{\partial}{\partial \tau} \ln[\phi_i(x_i, \tau)].$$

Since $n \geq 3$, we can choose $k \neq i, j$ and differentiate (4) with respect to x_i and x_k to obtain

$$\begin{aligned} \frac{\partial \psi_i}{\partial x_i}(x_i, t(\mathbf{x})) + \frac{\partial \psi_i}{\partial \tau}(x_i, t(\mathbf{x})) \phi_i(x_i, t(\mathbf{x})) &= \frac{\partial \psi_j}{\partial \tau}(x_j, t(\mathbf{x})) \phi_i(x_i, t(\mathbf{x})), \\ \frac{\partial \psi_i}{\partial \tau}(x_i, t(\mathbf{x})) \phi_k(x_k, t(\mathbf{x})) &= \frac{\partial \psi_j}{\partial \tau}(x_j, t(\mathbf{x})) \phi_k(x_k, t(\mathbf{x})). \end{aligned}$$

Since $\phi_k(x_k, t(\mathbf{x})) > 0$ by regularity, we deduce that

$$\frac{\partial \psi_i}{\partial x_i}(x_i, t(\mathbf{x})) = 0,$$

and therefore that there is a function $\overline{g_i}$ such that $\psi_i(x_i, t(\mathbf{x})) = \overline{g_i}(t(\mathbf{x}))$ for all $\mathbf{x} \in \mathbf{X}$. Choose τ_0 arbitrarily in $t(\mathbf{X})$ and, for any $\tau \in t(\mathbf{X})$, define

$$\overline{G}_i(\tau) = \exp \int_{\tau_0}^{\tau} \overline{g_i}(\tau')\, d\tau'.$$

Then, we have shown that, for all $j \neq i$,

$$\frac{\partial}{\partial x_j} \left\{ \ln[\phi_i(x_i, t(\mathbf{x}))] - \overline{G}_i(t(\mathbf{x})) \right\} = 0$$

and therefore the term in braces is a function of x_i alone, say $\overline{f_i}(x_i)$.

Hence, for all $\mathbf{x} \in \mathbf{X}$,

$$\ln[\phi_i(x_i, t(\mathbf{x}))] = \overline{G}_i(t(\mathbf{x})) + \overline{f_i}(x_i)$$

and therefore

$$\phi_i\left(x_i, t\left(\mathbf{x}\right)\right) = \widehat{f}_i\left(x_i\right) g_i\left(t\left(\mathbf{x}\right)\right),$$

where

$$\widehat{f}_i\left(x_i\right) = \exp\left[\overline{f_i}\left(x_i\right)\right] \text{ and } g_i\left(\tau\right) = \exp\left[\overline{G_i}\left(\tau\right)\right]. \tag{5}$$

Now observe that $g_i\left(\cdot\right) = g_j\left(\cdot\right)$ for all i and j since, substituting for ϕ_i in (3) and (4), we have

$$\widehat{f}_i\left(x_i\right) g_i'\left(t\left(\mathbf{x}\right)\right) \widehat{f}_j\left(x_j\right) g_j\left(t\left(\mathbf{x}\right)\right) = \widehat{f}_j\left(x_j\right) g_j'\left(t\left(\mathbf{x}\right)\right) \widehat{f}_i\left(x_i\right) g_i\left(t\left(\mathbf{x}\right)\right)$$

for all $\mathbf{x} \in \mathbf{X}$. This implies that

$$\frac{d}{d\tau}\left\{\overline{G_i}\left(\tau\right) - \overline{G_j}\left(\tau\right)\right\} = 0,$$

for all $\tau \in t\left(\mathbf{X}\right)$. Hence, $\overline{G_i}\left(\tau\right) - \overline{G_j}\left(\tau\right)$ is a constant whose value depends on i and j but not on τ. This implies the existence of $\alpha_1, \ldots, \alpha_n > 0$ such that $g_j\left(\tau\right) = \alpha_j g_1\left(\tau\right)$, for all j. Writing f_i for $\alpha_i \widehat{f}_i$, we conclude that ϕ_i has the form

$$\phi_i\left(x_i, t\left(\mathbf{x}\right)\right) = f_i\left(x_i\right) g_1\left(t\left(\mathbf{x}\right)\right),$$

for all $\mathbf{x} \in \mathbf{X}$. It follows from the definitions in (5) that both f_i and g_1 are positive.

Finally, choose τ_1 arbitrarily in $t\left(\mathbf{X}\right)$ and, for any $\tau \in t\left(\mathbf{X}\right)$, define

$$H\left(\tau\right) = \int_{\tau_1}^{\tau} \frac{1}{g_1\left(\tau'\right)} d\tau'$$

and note that H is a strictly increasing function. Writing $T\left(\mathbf{x}\right) = H\left[t\left(\mathbf{x}\right)\right]$, we have

$$\frac{\partial T\left(\mathbf{x}\right)}{\partial x_i} = H'\left[t\left(\mathbf{x}\right)\right] \frac{\partial t\left(\mathbf{x}\right)}{\partial x_i} = \frac{1}{g_1\left(t\left(\mathbf{x}\right)\right)} \phi_i\left(x_i, t\left(\mathbf{x}\right)\right) = f_i\left(x_i\right), \tag{6}$$

which implies

$$\frac{\partial^2 T\left(\mathbf{x}\right)}{\partial x_i \partial x_j} = 0 \text{ for } i \neq j,$$

for all $\mathbf{x} \in \mathbf{X}$. We may conclude that T is additively separable. So t can be written as the term in brackets in (1) where (6) implies that $F_i'\left(x_i\right) = f_i\left(x_i\right) > 0$ for all $x_i \in X_i$ and this means that F_i is a strictly increasing function. ■

The restriction to $n \geq 3$ is essential[2]. For example, if $n = 2$, $X_1 = X_2 = \mathbb{R}_{++}$ and $t = x_1^2 + x_1 x_2 + x_2^2$, then the hypothesis of the proposition is satisfied, since t is obviously regular and

$$\frac{\partial t}{\partial x_i} = \frac{3x_i + \sqrt{4t - 3x_i^2}}{2} > 0$$

for $i = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}_{++}^2$. If t were to satisfy (1), there would exist functions H, F_1, F_2 such that

$$H\left(x_1^2 + x_1 x_2 + x_2^2\right) = F_1(x_1) + F_2(x_2) \tag{7}$$

for all $(x_1, x_2) \in \mathbb{R}_{++}^2$. Differentiating this expression with respect first to x_1, then x_2 and eliminating $H'\left(x_1^2 + x_1 x_2 + x_2^2\right)$ gives

$$(x_1 + 2x_2) F_1'(x_1) = (2x_1 + x_2) F_2'(x_2). \tag{8}$$

Differentiating (8) with respect to x_1, using (8) again to eliminate $F_2'(x_2)$ and rearranging gives

$$\frac{F_1''(x_1)}{F_1'(x_1)} = \frac{3x_2}{(2x_1 + x_2)(x_1 + 2x_2)}.$$

Since this should hold for all $(x_1, x_2) \in \mathbb{R}_{++}^2$, we have a contradiction, so there are no H, F_1, F_2 for which (7) is satisfied.

3 A strategic equivalence

Given an aggregative game Γ in which the aggregator takes the form (1), we can define a new game Γ^* with the same player set in which player i now has strategy set $Y_i = F(X_i)$. For any $\mathbf{y} \in \mathbf{Y} = \prod_{j=1}^{n} y_j$ we shall write $Y = \sum_{j=1}^{n} y_j$ and let the payoff of player i be

$$\nu_i^*(y_i, Y) = \nu_i\left(F_i^{-1}(y_i), H^{-1}(Y)\right).$$

Then, the games Γ and Γ^* are strategically equivalent. We conclude that fully aggregative games with regular aggregators are strategically equivalent to a game in which the aggregate is a simple sum. For example, consider a simple n-player Tullock contest[3] with $X_i = \mathbb{R}_{++}$ and payoff

$$\frac{x_i^{r_i}}{\sum_{j=1}^{n} x_j^{r_j}} R - x_i,$$

[2] We are grateful to helpful observations on this matter by Martin Jensen.

[3] Such contests were first formulated by Tullock [18]. The large literature on contests is usefully surveyed in Konrad [10].

where $r_1, \ldots, r_n, R > 0$. Writing $y_i = x_i^{r_i}$ for all i shows that such a contest is strategically equivalent to a Cournot game in which inverse demand is unit-elastic: $p(Y) = R/Y$ and firm i's cost function is $c_i(y) = y^{1/r_i}$.

4 Applications

The family of games for which the aggregator function takes the form (1) includes several significant applications. Cornes and Hartley ([5], [6]) analyze Tullock contests with a general technology, and exploit the replacement correspondence approach to explore the behavior of equilibrium in contests involving many heterogeneous players. Cornes and Hartley [8] consider a family of public good models in which the aggregator function, instead of being an unweighted sum, takes a substantially more general additively separable form, and are able extend the analysis significantly beyond that provided by Cornes [3]. Finally, the idea of transforming a model of imperfect competition into a strategically equivalent oligopoly game with a homogeneous output was suggested by Spence [17] and discussed further by Yarrow [21]. However, those authors did not have the advantages of the replacement correspondence available to them to exploit fully Spence's insight. There seems to be promising scope for the application of our approach to various topics in industrial organization, environmental economics, and elsewhere within economics. By virtue of avoiding what Richard Bellman once called the curse of dimensionality, the replacement correspondence approach is particularly well-suited for the analysis of games involving many heterogeneous players.

References

[1] Corchon, L. (1994), Comparative statics for aggregative games: The strong concavity case, *Mathematical Social Sciences*, 28, 151-65.

[2] Corchon, L. (1996), *Theories of Imperfectly Competitive Markets*, Springer-Verlag.

[3] Cornes, R. C. (1993), Dyke maintenance and other stories: Some neglected types of public goods, *Quarterly Journal of Economics*, 107, 259 - 71.

[4] Cornes, R. C. and R. Hartley (2002), Joint production games with mixed sharing rules, Keele Economics Research Papers 2002/16, UK.

[5] Cornes, R. C. and Hartley, R. (2003), Risk aversion, heterogeneity and contests, *Public Choice,* **117**, pp 1–25.

[6] Cornes, R. C. and Hartley, R. (2005), Asymmetric contests with general technologies, *Economic Theory,* **26**, pp 923–946.

[7] Cornes, R. and R. Hartley (2007a), Aggregative public good games, *Journal of Public Economic Theory*, 9, 201-19.

[8] Cornes, R. and R. Hartley (2007b), Weak links, good shots and other public good games: Building on BBV, *Journal of Public Economics*, 91, 1684-1707.

[9] Friedman, J. (1982), Oligopoly theory, in Arrow, K. J. and Intriligator, M. (eds.) *Handbook of Mathematical Economics*, Volume I, North-Holland.

[10] Konrad, K. (2009), *Strategy and Dynamics in Contests*, Oxford University Press.

[11] Kukushkin, N. S. (1994), A fixed-point theorem for decreasing mappings, *Economics Letters,* 46, 23-26.

[12] Novshek, W. (1985), On the existence of Cournot equilibrium, *Review of Economic Studies*, 52, 85 - 98.

[13] Novshek, W. and H. Sonnenschein (1978), Cournot and Walras equilibrium, *Journal of Economic Theory*, 19, 223-66.

[14] Okuguchi, K. (1993), Unified approach to Cournot models: Oligopoly, taxation and aggregate provision of a pure public good, *European Journal of Political Economy*, 9, 233 - 45.

[15] Phlips, L. (1995), *Competition Policy: A Game-theoretic Perspective*, Cambridge University Press.

[16] Selten, R.(1970), *Preispolitik der Mehrproduktenunternehmung in der Statischen Theorie*, Springer-Verlag.

[17] Spence, A. M. (1980), Notes on advertising, economies of scale, and entry barriers, *Quarterly Journal of Economics*, 94, 493-507.

[18] Tullock, G. (1980), Efficient rent seeking, in Buchanan,J. M., Tollison, R. D. and Tullock, G. (eds.), *Toward a Theory of the Rent-seeking Society*, 131 - 46.

[19] Vives, X. (1999), *Oligopoly Pricing: Old Ideas and New Tools*, MIT Press.

[20] Wolfstetter, E. (1999), *Advanced Topics in Microeconomics*, Cambridge University Press.

[21] Yarrow, G. (1985), Welfare losses in oligopoly and monopolistic competition, *Journal of Industrial Economics*, 33, 515-29.